Teaching the principles of radioactive dating and population growth without calculus

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A dice playing game is proposed, by which means mathematically unsophisticated students and school children can learn empirically the principles of radioactive dating and population growth.

Most elementary astronomy courses describe in considerable detail how to measure astronomical masses and distances. The ideas behind these methods are understandable even to relatively young children, because no sophisticated mathematics is involved. Another quantity of astronomical importance is time; it is, in particular, rather important to know the age of the earth and of the solar system. However, the determination of the age of the earth and of meteorites is based on the behavior of radioactive materials as expressed by the Rutherford-Soddy decay law; this law is expressed either in exponential form or in terms of differentials, and both are generally incomprehensible to students who have not studied calculus. One often ends up mumbling vaguely about "measuring the radioactivity" of a piece of rock, which leaves the student no wiser about what is really done, and why. This is a pity, since the results are of fundamental importance.

In this note I outline a "game" which anyone can play whatever their level of mathematical competence and which simulates the basic processes involved. A similar game can be used to introduce the principles of population dynamics the biology students.

The Rutherford-Soddy law

$$dN = -Ndt/\tau$$

states that, in a group of N atoms, the fraction that decay dN/N, is a function only of time and of a constant τ, which is different for each type of atom. In other words, the probability that a given atom will decay is a constant.

This situation can be simulated by playing a game of chance. We first "prepare" a certain number of radioactive atoms. This can be just a sheet of paper ruled into squares: each square "is" a radioactive atom. The game then consists of throwing a die once for each atom; the result of the throw determines the fate of the atom. For example, one can impose the rule that a "6" indicates that the particular atom has disintegrated: we then put a cross through the square. We do this for all the atoms on the sheet; at the end of one such game the number of atoms remaining is noted. The game is then restarted, throwing the die for the "surviving" atoms only. At the end of each game, we note the number of surviving atoms, together with the ordinal number of the game. When no more atoms are left, a graph of the atoms remaining at the end of each game as a function of the ordinal number of the game will have the approximate shape of the Rutherford-Soddy law:

$$N_t = N_0 \exp(-t/\tau),$$

where N_t is the number of atoms remaining after time t, and N_0 is the initial number of atoms.

In this game, the ordinal number of a game represents a time interval, and the recipe chosen for eliminating atoms gives, in the example chosen, $\tau = 6$. By changing the recipe, one can generate different values of τ; we shall consider that, if the probability for "decay" is d,

$$\tau = 1/d.$$

Note that this method of relating the recipe to the decay constant τ involves a fundamental assumption: during a given time interval, the fractional number of atoms which "disappear" from the game is supposed to be a linear function of time. Now, for a finite game of chance, this is not strictly true; the Rutherford-Soddy law assumes infinitesimal time intervals (or very low decay probabilities). For a finite game involving large time intervals (or large decay probabilities), it would be more appropriate to work in terms of a survival probability $s = 1 - d$, rather than a decay constant τ; at the nth cycle of the game one will then have

$$N_n = N_0 s^n,$$

whence

$$N_t = N_0 s^t.$$

Comparing this relation with the "ideal" form of the Rutherford-Soddy law

$$N_t = N_0 \exp(-t/\tau),$$

we find that τ is related to the survival probability through

$$\tau = -1/\ln(s),$$

or to the decay probability through

$$\tau = -1/\ln(1-d).$$

However,

$$\ln(1-d) = -(d + d^2/2! + \cdots),$$

and for sufficiently small d,

$$\ln(1-d) \approx -d.$$

Therefore, in the limit of small d,

$$\tau \approx 1/d,$$

which is just what was assumed at the outset.

The game proposed in this paper is to be considered in the spirit of a model. Of course, no model can reproduce exactly a given phenomenon (otherwise, it would not be merely a model!); a model can represent a phenomenon only to a

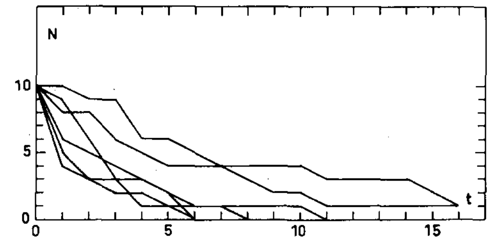

Fig. 1. Dice data for 10 initial atoms; $\tau = 3$. Six different "experiments" are shown.

certain precision, and is useful to the extent that deviations from the process we wish to simulate are smaller than other errors or uncertainties which arise. In the example given above, we are using the game to simulate a decay constant $\tau = 6$. The exact calculation gives $\tau = -1/\ln(1 - 1/6) \approx 5.5$. Our game is therefore a useful representation at the 10% level, which is quite adequate for the kind of statistics one might accumulate with dice; no one will have the patience to play at the 1% statistical fluctuation level. If someone does (for example, by using a programmable calculator—see the end of this paper), they can compare the results from games played with very large and very small decay probabilities and so study under which conditions the model approaches the process being studied.

Figure 1 shows the results of playing such games with 10 "atoms"; an atom was "struck off" when a throw gave a 5 or a 6, thereby simulating $\tau = 3$ ($\tau = 2.5$ with the exact calculation). Six different games were played—the results for all six are plotted in Fig. 1. One sees immediately a basic problem inherent to this type of simulation: purely statistical effects introduce rather strong fluctuations from game to game.

Figure 2 shows the result of averaging the six curves at each "time interval" (continuous curve): the fluctuations are much reduced and the curve now follows reasonably well the theoretical one computed for $\tau = 3$ (dotted curve). This highlights a pedagogically important feature of this game: individual players obtain widely scattered results, but a class average has the "right" properties.

If students encounter difficulties with averaging, which might raise the specter of fractional numbers of atoms, class results could first be added, before introducing the notion of averaging.

Figure 3 shows the results of playing the game with 50 radioactive atoms; the same recipe was used as before. Three different runs are plotted; the statistical fluctuations are of course much smaller and one can see clearly that the

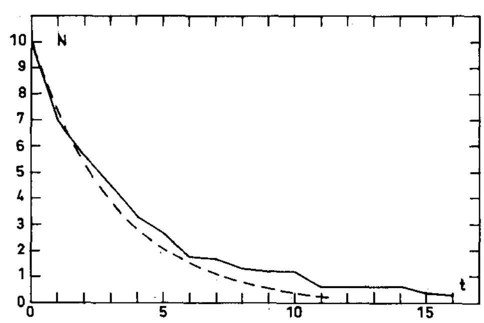

Fig. 2. The data of Fig. 1 averaged at each time interval. The Rutherford–Soddy Law with $\tau = 3$.

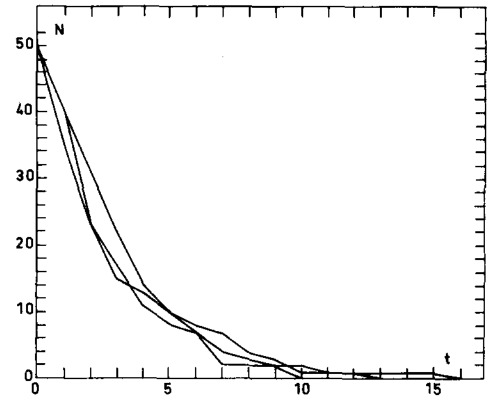

Fig. 3. Dice data for 50 initial atoms; $\tau = 3$.

"time" taken to reduce the number of atoms by ½ is certainly not proportional to the initial number of atoms (a remarkable result if one does not understand statistics too well). Of course, according to the Rutherford–Soddy law, the time taken to reduce the number of atoms by one-half, the half-life $t_{1/2}$, is related to τ through

$$t_{1/2} = -\tau \ln(0.5).$$

Figure 4 shows the results of changing the decay "recipe": an atom was struck off when a throw gave a "6" only, corresponding to $\tau = 6$. Three different runs for 50 starting atoms are shown; one can see that the half-life is effectively about twice as long as before.

One way of organizing a class to play this game efficiently might be the following:

Everyone is made to play the game several times with a small number of atoms (say 10) and to take the average of his own results. This is quickly done, and by comparing their individual averages, students can accustom themselves to accept each other's work and to take class averages.

Different groups or individuals can then be assigned different starting conditions (number of atoms, τ); the basic phenomena will then emerge from an intercomparison of individual results.

The principles of one type of radioactive dating can now be demonstrated. One "player" plays the game privately using one of the recipes for which a decay curve has already been obtained ("calibration"). He counts only the ordinal number of the game, and stops at some convenient point before the end. He then gives to the others the ratio of the number of decayed to undecayed atoms at the point where

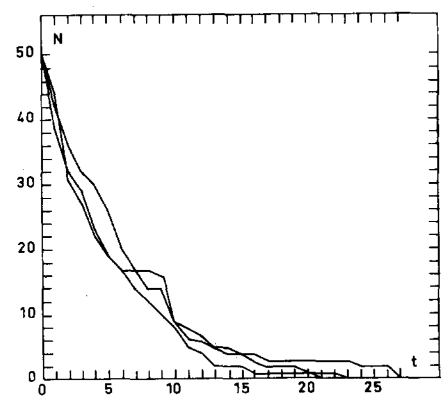

Fig. 4. Dice data for 50 initial atoms; $\tau = 6$.

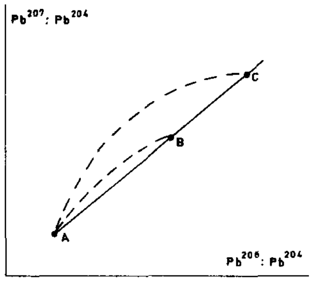

Fig. 5. Sketch of the lead isotope ratios at a given time from two samples of radioactive material created simultaneously. The slope of the line *AB* isochrone is a function of age. *A:* initial conditions; *B:* sample 1; *C:* sample 2. The dotted lines show the growth curves for each sample.

he stopped; his friends have to deduce how many games he played, and the probable error in the determination.

This game shows how one finds an age if the initial sample contained none of the product nucleus. Radioactive dating is somewhat more difficult if this cannot be guaranteed (for example, the Earth, meteorites, etc.); in this case, it is necessary to use the radioactive decay of two elements. The principle of one such method is quite simple (see, for example, Eicher[1]). Suppose we use a sample containing U^{238} (which decays to Pb^{206}) and U^{235} (which decays to Pb^{207}), for which the decay constants are rather different. At an initial instant of time, the sample contains U^{238}, U^{235}, Pb^{204}, Pb^{206}, and Pb^{207} in unknown ratios. We note that Pb^{204} is not a product of any radioactive decay. As the sample ages, the Pb^{206}:Pb^{207} ratio changes since the decay constants are different. Figure 5 shows the variation, plotted in terms of the Pb^{206}:Pb^{204} and Pb^{207}:Pb^{204} ratios since the quantity of Pb^{204} cannot change.

Point *A* represents the initial conditions, and point *B* the situation after a certain length of time. The slope of the line *AB* gives the age of the sample; however, with one sample, we cannot obtain the slope because the initial conditions are unknown.

Suppose we now have a second sample, created at the same time as the first. We suppose that the ratio Pb^{206}:Pb^{207}:Pb^{204} is as it was in the other sample (this will be fixed by whatever process created the elements in the first place); however, the ratio U:Pb need not be the same since this will be the result of mixing, crystallization, and so on. As before, as the sample ages, the ratio Pb^{206}:Pb^{207} changes; when the same amount of time has elapsed this ratio will have the same value as before. Consequently, in the Pb^{206}:Pb^{204} versus Pb^{207}:Pb^{204} plot, this sample will fall on the line *AB*, whose gradient can now be determined. In practice, one uses many meteorites so that the line is rather well defined. The line obtained is generally called an "isochrone."

The game we have outlined can be played for this case also: one simply has, for a given sample, two sheets of paper (for each of the elements U^{238} and U^{235}) into which one has already "mixed" a certain number of lead atoms. Different samples will have different starting ratios U:Pb (but the same Pb^{206}:Pb^{207}:Pb^{204} ratio) and one plays, for each sample, the same number of games (using, of course, different recipes for U^{238} and U^{235}). One then shows that the results fall on a line whose gradient gives the number of games played. With enough samples being played over, one can show how the accuracy of the result depends on the number of samples available.

Once again to save time and eliminate possible boredom, different groups of students can be assigned different "samples," so that an isochrone can be obtained quite quickly. Alternatively, one can use the dice throws already obtained for the previous "experiments", modifying only the basic "data sheet" used.

I have tried this game on a group of particularly gifted 11-year olds; the results were most satisfying. Two collegues also tried it out on a class of quite average 13–14-year olds and their reports where also favorable.

With this game we have of course reproduced what is called by physicists a "Monte Carlo simulation"; this kind of procedure can be very useful for studying the behavior of complicated systems. It is worth while pointing out to the student the value of such techniques, especially in an age of computers when Monte Carlo simulations can be carried out swiftly and for very complex problems where analytical solutions are not forthcoming. He might thereby better appreciate why his pocket calculator has (as some do) a random number generator. Indeed, many otherwise mathematically quite unsophisticated students have learnt to program pocket calculators with extraordinary ease; an interesting exercise would be to make such students simulate the entire game on a programmable calculator. I have done this myself and it is quite amusing. Incidentally, one must beware of pseudorandom number generators on some calculators, the distribution is far from random.

The discussion so far has turned around radioactive dating. However, one can think of this game as a means of showing students empirically one reason why the exponential function crops up in nature, and under which conditions random phenomena can give rise to systematic effects.[2] Another particularly relevant example would be population growth in biological systems the basic equation governing an uninhibited rise in population is just the Rutherford–Soddy law, without the minus sign. To simulate this, one might start with just one "member" of a population and play the game as before, this time *adding* members for each successful throw. One can make the game more "realistic" by including a criterion that members of the population start to "eat" each other above a certain critical population density. Plotting the results on log paper shows the biology student immediately why he always does this: uninhibited growth gives a straight line, while all other cases indicate the presence of underlying control mechanisms.

[1]D. L. Eicher, *Geologic Time*, Foundation of Earth Science Series (Prentice-Hall, Engelwood Cliffs, NJ, 1976).

[2]It would appear that two PSSC films represent a particularly useful adjunct to this paper. Living in France, I have had no occasion to see them, however, it might be useful for readers to be aware of their existence: *Random events* produced by Hume and Ivey, Toronto (distributed by Ward's Natural Science Establishment, Inc., P.O. Box 1712, Rochester, NY 14603) 16 mm. b/w, 31 min; *Long time intervals* produced by Brown, Cal. Tech. (distributed by Ward's Natural Science Establishment, Inc., P.O. Box 1712, Rochester, NY 14603) 16 mm. b/w. 25 min.