A simple way to assess the structure of red giants

Ludwik M. Celnikier
Observatoire de Paris—Meudon, 92195 Meudon Principal Cedex, France

(Received 14 July 1988; accepted for publication 24 March 1989)

A simple semianalytical calculation is used to study how a star reacts when its central stock of hydrogen is exhausted and before the next fusion reaction based on helium begins.

I. INTRODUCTION

The structure of homogeneous stars "kept alive" by the fusion of hydrogen into helium in a central region (the so-called "main sequence" phase) is a rather well-worn subject: Accessible explanations (as distinct from a full numerical integration of the nonlinear equations of stellar structure) vary from the purely verbal (see any popular book on astronomy) to quite sophisticated computational exercises[1] designed for a micro- or minicomputer, via different levels of physical understanding and analytical approximations (see, for example, Refs. 2–8). Reasonably convincing quantitative discussions exist also for the final stages of stellar evolution (see Refs. 1–9). The semianalytical accounts allow one to estimate the fundamental parameters of a star starting from the essential physics; the result is more or less accurate, depending on just what goes into the intellectual gymnastics.

This does not seem to be true of the intermediate stages of stellar evolution. We know that from the moment when a star has stabilized, it is using up its central stock of hydrogen so that the central "powerhouse" must eventually run short of available hydrogen to convert into helium. The way the central regions become exhausted depends on the mass of the star.

Stars whose central regions have some degree of convection and so are reasonably well mixed will tend to exhaust "in one fell swoop" the hydrogen in a finite volume around the center, thereby precipitating a relatively sudden crisis in which gravitational contraction remains temporarily the only viable source of the energy that the star continues to radiate; the contraction heats up the now "dormant" core, ultimately bringing it to a temperature at which helium can begin to fuse into heavier elements. However, before the new reactions begin, the so far unused hydrogen at the edge of the helium-rich volume is dragged willy nilly into hotter regions, where it begins to fuse and so release energy: For a time, the star can be maintained by the power produced by a "shell" burning hydrogen at a temperature that is not directly influenced by the general equilibrium conditions of the star as a whole.

The behavior of low-mass stars is rather different: It turns out in this case that the central regions are not well mixed so that hydrogen exhaustion occurs progressively, working outward from the center; the core adjusts gradually to its changed circumstances, contracting slowly and thus compensating for the power loss by bringing gradually into higher temperatures the hydrogen that lies beyond the active core. In this way, the star creates an active shell that gradually takes over the energy production from the dying core, but the process can be very slow. Nevertheless, at some very late stage, the star will have a dormant and compact hydrogen-exhausted core while power will be supplied by a surrounding shell.

It is generally accepted (see, for example, Ref. 2) that about 1.3 $M_\odot$ constitutes the borderline between these two broad classes of evolution. Although the details of the evolution are rather different, there is a certain convergence in the sense that qualitatively similar structures do emerge sooner (massive stars) or later (less massive stars).

The equilibrium state of such a structure—a dormant and very compact core surrounded by an active shell, itself surrounded by an envelope that transports the energy from the shell to the outside—is usually studied numerically since the equations governing such a problem are even more nonlinear than those which apply to homogeneous stars. Results from this type of calculation are given, for example, in Ref. 10.

However, numerical problems that appear in the integrations usually force one to carry out the computations small zone by small zone, with some continuity condition being imposed at the frontiers of the zones in order to obtain a coherent solution, and it is then very easy to lose sight of the essential physics of the problem, especially as the computations understandably bring into play not just one principal phenomenon but many, coupled together in inextricable ways.

Standard explanatory texts on the subject of red giants, as these structures are called, are at best very discrete, and some are frankly misleading; many sound like the verbal rendering of a computer printout, especially as few attempts are made to estimate the resulting stellar parameters (a notable exception being Reddish[11]) from what are presented as "obvious" steps. In fact, finding a physical interpretation for certain of the phenomena that emerge from the numerical computations is the subject of some discussion even among specialists, as one can see from the opening remarks of Yahil and Van den Horn[12] and the recent paper by Applegate.[13] Indeed, the latter paper, while quite properly raising the issue, illustrates how easy it is to lose sight of the essential physics when a powerful computer is available: The author solves numerically a purified stellar radiation transfer problem out to a distance of very many parsecs (which constitutes an interesting exercise, but one wonders if it is really necessary to let the integration run so far, since stars are rather *small* objects!), without coming to grips with the phenomena that force the stellar envelope to recede in such a remarkable way. A short critique of this paper has recently been published (Ref. 14).

The present contribution is an attempt to show how basic physics and a useful analytical trick can be combined to study a structure composed of a dormant core, an active shell and an envelope—a structure associated with the name "red giant." The idea is not at all to provide an alternative to numerical analyses (without which stellar evolution would not be a viable subject), but rather to furnish a little insight into how some of the processes articulate within an intrinsically complicated object; in this sense, the approach is similar to that used to understand the structure of homogeneous stars and in consequence suffers from the same type of liability—it is not clear to what *particular* object the model applies and it is not even sure that *any* star is quite like this. That, however, is the price paid to make the problem analytically tractable; does any planet, mountain, or star in fact *really* work the way that Weisskopf described in his classic papers?

Some material, well known to astrophysicists but perhaps not to others, is given in summary form. Unless otherwise stated, all units are SI; the symbol $\odot$ is used to identify the Sun.

II. THE VIRIAL THEOREM

The virial theorem expresses the equilibrium configuration of a mechanical system; it can be applied to a mass of gas in equilibrium under the opposing forces of its self-gravitation, internal pressure, and externally applied pressure and the result is discussed in most standard textbooks on mechanics or astrophysics. A particularly convenient form is derived by Chiu[15]:

$$-3\int V\,dP = U_g, \qquad (1)$$

where V and P are the volume and pressure within the gaseous sphere, and U_g is its total self-gravitational energy.

The left-hand side can be integrated by parts, which gives after a little rearrangement,

$$4\pi r^3 P_{\text{ext}} = 3\int P\,dV - U_g,$$

where P_{ext} is the external pressure acting on the surface of the mass of gas.

The next part of the discussion is also quite standard (see, for example, Ref. 16) but is more often than not elided from many recent discussions, thereby confusing the issue (at least for massive stars) in a quite unnecessary way.

Let us simplify the previous expression by using an average temperature $\langle T\rangle$ and by assuming that the sphere has uniform density; this gives

$$4\pi r^3 P_{\text{ext}} = 3\,\frac{k\langle T\rangle M}{\mu} - \frac{3}{5}\,\frac{GM^2}{r}, \qquad (2)$$

where M is the total mass of the sphere, r is its radius, and μ is the atomic mass of the gas.

This expression can be applied to any internal region of radius r_c, mass M_c, and average temperature T_c; P_{ext} is then the pressure that the rest of the star applies at r_c. Now, while a star in equilibrium is burning hydrogen in a central volume, its central temperature is a relatively invariant quantity, dependent essentially on the stellar mass. This is fully explained in elementary texts and has an important consequence: For a given central mass of a given star with a given μ, P_{ext} is not a monotonic function of r as it is the difference between two terms whose functional dependence on r_c is different:

$$P_{\text{ext}} = a/r_c^3 - b/r_c^4.$$

This type of equation has an extremum in r_c, which a little analysis shows to be a maximum and which may be located in the usual way:

$$P_c^{\max} \approx 3\,\frac{1}{G^3}\,\frac{1}{M_c^2}\left(\frac{kT_c}{\mu}\right)^4. \qquad (3)$$

We have thus obtained an expression for the maximum external pressure that the central region of a star can sustain without change in temperature.

Equation (3) is of only limited interest for a star having uniform chemical composition since it tells us little more than one might conclude from a little reflection: The larger M_c and the lower T_c, the lower is the allowed maximum pressure, which is just what one might expect since this corresponds to regions closer and closer to the stellar surface.

However, the maximum pressure as given by Eq. (3) is a very sensitive function of the atomic mass μ. A change by just a factor 2, corresponding to a change from "normal" stellar composition to one in which the hydrogen has been transformed to helium, lowers the maximum allowed pressure by over an order of magnitude: If a sufficient quantity of hydrogen in the central regions of a star has been converted to helium, that section becomes unstable if its temperature does not rise. In essence, the conversion of hydrogen to helium has lowered the number of particles that contribute to the outwardly acting pressure, which therefore drops if the temperature does not change; however, the temperature can only rise if the unstable mass contracts.

This result, obtained here heuristically and whose complete form is known as the Schönberg–Chandrasekhar limit, imposes an upper limit on the mass of a hydrogen-exhausted stellar core that can remain in equilibrium at a given temperature within a larger hydrogen-rich structure; it highlights the rather special role played by the "dormant" core that is subject to a *local* instability leading to a *local* contraction...but nothing more.

For a Schönberg–Chandresekhar type of instability to occur, sufficient hydrogen should have been converted to helium without significant changes occurring in the other parameters. This will be the case in massive stars, where central convection is continuously mixing spent and unspent material so that when hydrogen exhaustion does occur it does so over a relatively important volume quite suddenly. This will be less true in low-mass stars, where the exhaustion is gradual from the center outward and the core contracts gradually: In spite of this, a time will arrive when even low-mass stars will be unable to generate energy in a significant fraction of their core region, and will not yet have reached temperatures such that a helium reaction can function.

The quantitative part of the analysis is made with reference to an object of $1.3\times M_\odot$, since this is the minimum mass for which one can (with more detailed analysis) identify an independent phase of helium-rich core contraction; a numerical model for just such a star with a condensed dormant core containing 26% of the total mass is given in tabular form by Schwarzschild (Ref. 2, Table 28.7), so that one may compare the results of our simple estimates with a more conventional approach. Note, however, that this model has already been a red giant for several billion years, and so its evolutionary phase is quite distinct from that of the model we shall estimate. The parameters of the initial state, when hydrogen was being "burnt" in the core, are unfortunately not available in this book for this particular mass; they may be estimated roughly using the techniques described by Celnikier,[8] remembering, however, that convection does take place in the core region of a $1.3\,M_\odot$ star, which the model ignores. Schwarzschild's[2] Table 28.6 is a model of the present Sun, and one may use it as a guide to the initial structure of a 1.3 solar mass object—this should suffice for comparisons with the kind of simple estimates made below.

The analysis that follows investigates the situation just outside the initial helium core, a region that is now at the edge of the red giant model's core and so constitutes a zone whose parameters can be compared from model to model; it is still within the hydrogen-burning core of the original main sequence model. We will assume that initially the hydrogen-exhausted core occupied the same volume as when it was generating energy, extending therefore (see Schwarzschild's[2] table for a solar mass object) to about 0.2 of the original radius. This is, of course, a gross caricature, equivalent to assuming that the energy-producing process suddenly switches off; however, the detailed path by which the star actually reaches the state tabulated in Schwarzschild's[2] Table 28.7 is not here the subject under study, only what the star will be like when it gets there. Nor are we investigating the time scales involved, which will be very different for low- and high-mass stars.

The existence of a local instability suggests dividing the star into three distinct regions: the helium-rich core, the hydrogen-rich envelope, and a transition zone, which, we shall see, is where all the action takes place.

III. THE HYDROGEN-EXHAUSTED CORE

As discussed above, the hydrogen-exhausted core must shrink and in the process must heat up (virial theorem *oblige*), until conditions are such that it can support the external pressure.

How does the core evolve during its contraction?

Since it is no longer a source of thermonuclear energy, one expects that over a period of time it will tend to become isothermal since a temperature gradient cannot be maintained without a heat flow and a heat flow requires a heat source; however, in itself this change in the temperature distribution is of minor consequence since the inner 20% to 30% of the mass of a sunlike star is in any case almost isothermal—over such a region the temperature changes by a factor of at most about 2. For a perfect gas at temperature T and density ρ,

$$P \propto \rho T$$

so that

$$\Delta P \propto \Delta\rho\, T + \Delta T\, \rho$$

or

$$\frac{\Delta P}{P} = \frac{\Delta \rho}{\rho} + \frac{\Delta T}{T}.$$

Substituting the conditions relevant, for example, to a solar mass star, we see immediately that the temperature gradient contributes to the pressure difference across the central zone as much as, but not significantly more than, the density gradient. Therefore, a mere doubling of the density gradient can compensate for a complete disappearance of the temperature gradient. This point is worth emphasizing since many texts suggest (without proof) that the development of an isothermal core is alone responsible for the appearance of an important density gradient: While the density gradient does in fact steepen considerably in the hydrogen-exhausted stellar core, whose temperature distribution does tend to approximate to an isothermal one, it is misleading to attribute this steepening exclusively to the change in temperature distribution.

Note that since the core is actually contracting and therefore evacuating gravitational energy, its temperature distribution cannot ever be truly isothermal; on the other hand, contraction eventually renders the electrons partially degenerate and so raises considerably the thermal conductivity of the core region. Overall, one does not expect departures from an approximately isothermal state in the core to be important, and we use this without rigorous proof to facilitate the computations; a more detailed discussion can be found in Ref. 2.

A. Temperature rise of the hydrogen-exhausted core

The contraction of the core is governed, as we have seen, by Eq. (2); if we assume that the core does remain isothermal during this phase and that the external pressure acting on it does not change appreciably (see below), the ratio of the temperature T_c after contraction to that before, $T_{\rm orig}$, may be written as

$$\frac{T_c}{T_{\rm orig}} \approx \frac{r_{\rm orig}}{r_c}\left(\frac{4\pi P r_c^4 + 3GM_c^2/5}{4\pi P r_{\rm orig}^4 + 3GM_c^2/5}\right), \qquad (4)$$

where M_c is the mass of the core, $r_{\rm orig}$ is its radius before collapse when the temperature was $T_{\rm orig}$, and r_c is its radius at some stage during the contraction when the temperature has risen to T_c.

We have taken the exhausted helium-rich core to comprise 26% of the mass, as in Schwarzschild's[2] table. What values of $r_{\rm orig}$ and P may be plausibly substituted in Eq. (4)?

It is easy to show using the approximate analytical results by Celnikier[8] that the radius of a 1.3 $M_\odot$ homogeneous star is only marginally different from that of the Sun, that 26% of the mass is contained within a radius equal to 0.21 of the stellar radius where the pressure is 5.7×10^{15} Pa, and that the average temperature within such a volume is just under 1.4×10^7 K. Moreover, this volume is responsible for 80% of the stellar luminosity.

In Schwarzschild's model,[2] the core has shrunk to a radius of 1.62×10^7 m; applying these parameters to Eq. (4), one finds

$$T_c = 3.8\times 10^7\ \text{K},$$

a value that can be compared to the 4×10^7 given by Schwarzschild.[2]

This degree of agreement, while highly satisfactory, should perhaps not be taken too seriously. The "starting" core parameters relate to an approximate model; the model turns out to give a reasonable description of the Sun (see Ref. 8), but it does ignore core convection, which is a feature of more massive stars. Even in the case of the Sun, the radius is somewhat underestimated: I am asserting simply that since the model predicts practically the same radius for a 1.3 $M_\odot$ star as it does for a 1 $M_\odot$ object, one can use with reasonable safety the actual solar radius for the radius of an actual 1.3 $M_\odot$ star.

One might also do well to recall that electron degeneracy sets in gradually during the contraction of the core—one can in principle take account of this, but it seems hardly worthwhile in the present context; in fact, the calculation being made here would be more correct for massive stars, since their central densities are lower (see, for example, Ref. 8) and so they are farther from degeneracy. However, numerical models for the advanced evolutionary phases of such stars are not as easily available as the case being treated here, and in particular are not given in the book by Schwarzschild.[2]

Note, finally, that the factor $\frac{3}{5}$ that appears in the expression for the gravitational energy applies in principle only to a sphere of uniform density; while justifiable for the initial state, one should use a different factor for the final state since an important density contrast does develop between the center and the edge of the collapsing core (see Sec. III B). This refinement also seems hardly worthwhile here.

B. The density of the hydrogen-exhausted core

The importance of the density variation in the core may be assessed using the method of trial functions pioneered by Acton and Squire[17]; in essence, an approximate solution to many tricky boundary value problems can be found by substituting into the model equations simple functions whose overall behavior mimics that of the expected exact solution and which are expressed in terms of parameters whose values we wish to know. Physical intuition, luck, or just plain black magic is used to identify appropriate and convenient functions; the parameters are found by forcing the trial functions to satisfy the model equations at some "typical" value of the independent variable.

The method was applied by Celnikier[8] to find analytical expressions for the global structure of a star.

The contraction of the core cannot proceed faster than energy is evacuated from it and so is relatively slow, slower than the speed at which mechanical waves propagate in the gas. Consequently, one can assimilate its evolution to a sequence of "stationary" states, each of which can be computed as if it were in hydrostatic equilibrium. The pressure P, density ρ, and mass M distribution within the core will thus be the solutions of the following two equations:

$$\frac{dP}{dr} = -\frac{\rho GM}{r^2}, \tag{5}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho, \tag{6}$$

to which must be added an equation of state, which we take to be that of an ideal gas

$$P = \alpha(kT_c\rho/m_p), \tag{7}$$

where α is approximately equal to 0.76 if the chemical composition is dominated by helium and 1.65 in the case of the "standard" interstellar mix of hydrogen and helium; m_p is the mass of the proton and T_c is the core temperature, assumed to be uniform throughout the core.

Note that in Eq. (5) one should in principle use the total pressure, which is the sum of the gaseous pressure given by Eq. (7) and the radiation pressure: The radiation pressure is in fact a small fraction of the total, and will be ignored here.

We know that the density decreases from the center to the outside while the mass increases; we know also that their first derivatives vanish at the center. A convenient trial function for the variation of the core mass is

$$M = 2M_c(1 - \{1/[1 + (r/r_c)^3]\}), \tag{8}$$

where r_c and M_c are the radius and mass of the core; this expression ensures a positive definite value for the mass and a clear value for the core mass at a particular value of the core radius. Note that this expression is being used to model the core mass only: It is in no way implied that the total mass of the star tends to twice the core mass as $r \to \infty$.

For the density distribution we can take

$$\rho = \rho_c \exp(-r^2/\lambda_\rho^2), \tag{9}$$

where ρ_c is the central density and λ_ρ is a parameter that gives a measure of the spatial density distribution. This expression ensures that the density never becomes negative.

Most of the art in the method of trial functions resides in the choice of appropriate functions. It is just as well to emphasize that their exact form has no great significance since they are being used only as mathematical supports to obtain an approximate analytical solution to an otherwise analytically insoluble boundary value problem; the choice of functions is constrained by known boundary conditions and the functions should be consistent with the general behavior of the model equations.

These trial functions, together with the ideal gas equation (7), are substituted into Eqs. (5) and (6), which gives

$$\frac{r}{\lambda_\rho^2} = \frac{GM_c m_p}{\alpha k T r^2}\left(1 - \frac{1}{1 + (r/r_c)^3}\right), \tag{10}$$

$$\frac{3M_c}{r_c^3}\frac{1}{[1 + (r/r_c)^3]^2} = 2\pi\rho_c \exp\left(-\frac{r^2}{\lambda_\rho^2}\right). \tag{11}$$

These equations are, of course, not exact and will not be satisfied for an arbitrarily chosen value of r; however, the right-hand sides, for example, have a globally similar behavior to the left-hand sides of the corresponding equations and, in the spirit of Acton and Squire's method,[17] the expressions are forcibly satisfied ("collocated" in the language of this method) at some intermediate point in the r range, which we will plausibly take at $M_c/2$, i.e., at $r_c/r = 3^{1/3} = 0.694$. After some straightforward manipulation, one obtains

$$\left(\frac{r_c}{\lambda_\rho}\right)^2 = \frac{3}{4}\left(\frac{GM_c m_p}{kT_c r_c \alpha}\right), \tag{12}$$

$$\rho_c = \frac{3M_c \exp(0.69^2 r_c^2/\lambda_\rho^2)}{2\pi \times 1.333^2 \times r_c^3}. \tag{13}$$

Putting α equal to 0.76 (hydrogen-exhausted core) and $T_c \approx 3.8 \times 10^7$ (Sec. III A), one finds

$$(r_c/\lambda_\rho)^2 \approx 8.72,$$

which corresponds to a density ratio between the edge and the center of the core equal to 1.7×10^{-4}. This ratio is much smaller than in the case of a hydrogen-burning main sequence star: This comes about essentially because the average density of the core is now very high, leading to a relatively small density scale height (see Sec. IV) and so to a relatively rapid variation of the core density with distance from the center.

From Eq. (13), the central density evaluates to 2.7×10^9 kg m^{-3}, and so the density at the edge of the core is 4.4×10^5.

Finally, in this simplified treatment, the chemical composition changes abruptly from one that is helium dominated to one containing a mixture of hydrogen and helium; since the pressure and temperature must be continuous, the density just outside the core, on the hydrogen side, must drop to ${}^{\mathrm{H}}\rho_c$,

$$ {}^{\mathrm{H}}\rho_c = 4.4\times10^5(0.76/1.65) \tag{14}$$

$$ = 2\times10^5 \text{ kg m}^{-3}. \tag{15}$$

The density at the center and (hydrogen side) edge of the core in Schwarzschild's[2] model is 3.5×10^8 and 6.9×10^4 kg m^{-3}, respectively.

Knowing the density and temperature at the edge of the core, we can estimate the pressure, which comes to $\approx 6\times10^{16}$ Pa; this estimate of the pressure can be used to improve the estimate of the central temperature by substituting it back in the numerator of Eq. (4) and repeating all the calculations: One finds that the results are virtually unaltered, showing that the computation is at least internally consistent.

IV. THE INTERFACE BETWEEN THE SHRINKING CORE AND THE REST OF THE STAR

The hydrogen-exhausted core has become compact and dense. If we assume that just outside the core edge dT/dr is not of overriding importance (see below), Eq. (5) can be written in the form

$$d\rho/\rho = -(GM_pM_c/\alpha kT_c r_c^2)dr,$$

which can be integrated immediately to give the familiar equation

$$\rho = {}^{\mathrm{H}}\rho_c e^{-h/\eta}, \tag{16}$$

where h is the height above the core edge and η is the so-called "scale height," given by

$$\eta = r_c^2\alpha kT_c/m_pGM_c. \tag{17}$$

The scale height on the hydrogen side of the core edge is $\approx 3\times10^6$ m, which is an order of magnitude smaller than the core radius itself and about 20 times smaller than the scale height above the dormant core was before contraction began. Therefore, the density will fall very rapidly in the immediate vicinity of the core edge; this very steep density gradient is a direct consequence of the high density of a small core, as one can see from Eq. (17).

To show that for the purpose of estimating the scale height just outside the hydrogen-exhausted core (and whose exact value is in any case not important) we may ignore the contribution of dT/dr, it is sufficient to substitute the ideal gas equation in Eq. (5); after a little rearrangement, this gives

$$\frac{d\rho}{dr} = -\frac{{}^{\mathrm{H}}\rho_c}{T_c}\left(\frac{Gm_pM_c}{\alpha kr^2} + \frac{dT}{dr}\right).$$

The first term within the parentheses on the right-hand side evaluates to about 30; in Sec. IV A we shall see that the average value of dT/dr in the burning shell is about 40. Even taking account of the temperature gradient would not change the scale height by a huge factor. We shall see below that in any case the scale height is needed only for qualitative judgments and its exact value has no impact on our conclusions.

As we have seen, the temperature on the hydrogen side of the core edge is now over three times higher than it was before contraction began; consequently, hydrogen fusion will be initiated in a surrounding shell.

A. The hydrogen-burning shell

Hydrogen fusion in stars proceeds via two principal reactions, the "*p-p*" and "CN" cycles. The former dominates at temperatures close to 10^7 deg; the latter at higher temperatures. It is therefore clear that at the temperature associated with the edge of the contracting stellar core, energy will be generated by the "CN" cycle, for whose rate ϵ Schwarzschild[2] quotes a convenient power law approximation,

$$\epsilon = 10^{-\mu}(XX_{\mathrm{CN}})\rho T^{\nu}, \tag{18}$$

where X and X_{CN} are, respectively, the fractional abundances of hydrogen and elements such as carbon and nitrogen; taking Schwarzschild's[2] "standard" values for interstellar material, the product XX_{CN} is equal to 4.7×10^{-3}. The values of the coefficients μ and ν depend on the temperature range; at around 3.8×10^7 K, the values of the coefficients are (in SI)

$$\mu = 97.5,$$

$$\nu = 13,$$

the temperature T being written directly in Kelvin.

The power output L ("luminosity") of the active shell is given by

$$\frac{dL}{dr} = 4\pi r^2\epsilon\rho \tag{19}$$

$$= 4\pi r^2(XX_{\mathrm{CN}})\times10^{-\mu}\times\rho^2T^{\nu}, \tag{20}$$

whence

$$L = \int_{r_c}^{r_{\max}} 4\pi r^2(XX_{\mathrm{CN}})\times10^{-\mu}\times\rho^2T^{\nu}\,dr. \tag{21}$$

Now, ν is much greater than 2, from which it follows that the power output is particularly sensitive to the value of the temperature; since the temperature is a decreasing function of r, one might well expect the main contribution to L to come from a very small range in r around the core edge.

Let us assume as a working hypothesis that the range in r over which L is important is so small that to evaluate the integral in Eq. (21) we may put without significant error

$$r = r_c,$$

$$\rho = {}^{\mathrm{H}}\rho_c.$$

This will be justified *a posteriori*; in the meantime, we can express the variation in luminosity as a function of the altitude h above the core edge, so that

$$\frac{dL}{dh} = \alpha T^{\nu}(h), \tag{22}$$

where α is a constant that can be computed in terms of known quantities:

$$\alpha = 4\pi r^2(XX_{\mathrm{CN}})\times10^{-\mu}\times({}^{\mathrm{H}}\rho_c)^2.$$

Let us now represent the change of T through the shell by a linear law of the type

$$T = T_c(1 - mh/T_c), \tag{23}$$

where m is an average thermal gradient and h is again the height above the core edge. This is, of course, a convenient fiction: The relative temperature gradient will tend to zero at the inner edge of the shell (where the energy generation vanishes) and thereafter rise to some limiting value at the outer edge (corresponding to the generated luminosity). Substituting Eq. (23) in Eq. (22) and rearranging,

$$L = \alpha T_c^{\nu} \int_0^{h_{\max}} \left(1 - \frac{mh}{T_c}\right)^{\nu} dh.$$

Putting

$$u = 1 - mh/T_c, \tag{24}$$

one obtains finally

$$L = -\{\alpha T_c^{\nu+1}/m(\nu+1)\}[u^{\nu+1}]_1^{u_{\min}}. \tag{25}$$

In this expression, $u_{\min}$ is the value of u that corresponds to $h_{\max}$; although in principle one does not know how far out the integration limit should be, it is clear, since $\nu = 13$, that once $u_{\min}$ has dropped to, say 0.85, more than 90% of the luminosity has been accounted for. This fixes the effective width $h_{\max}$ of the hydrogen-burning shell.

Now it is shown in standard texts that a certain temperature gradient dT/dr is needed in order to "drive" a power L across a stellar shell; if the energy is being transported entirely by radiation transfer,

$$\frac{dT}{dr} = -\frac{3}{64\pi\sigma} \frac{L\kappa\rho}{T^3 r^2}, \tag{26}$$

where κ is called the opacity, and is such that:

(a) If electrons only are responsible, κ is independent of temperature, and has the value 0.034.

(b) If partially ionized atoms are the main source of opacity, one may use Kramer's law, for which

$$\kappa = 7\times 10^{20} Z(1+X)\rho T^{-3.5}, \tag{27}$$

where X and Z are the fractional abundances of hydrogen and elements heavier than helium, respectively; the product $Z(1+X)$ is about 0.034.

At the edge of the hydrogen-burning shell, Eq. (27) gives 0.03: Electron scattering furnishes as important a contribution to the opacity as scattering from heavy ions.

Expression (25) contains the average thermal gradient m; in the spirit of this approximation, the average gradient will be the characteristic value of dT/dr over the shell, which we may obtain from Eq. (26) by substituting characteristic values

$$m = \left\langle \frac{dT}{dr} \right\rangle$$
$$= -(3/64\pi\sigma)\,(L\kappa\langle\rho\rangle/\langle T\rangle^3\langle r\rangle^2). \tag{28}$$

In the spirit of a rough evaluation, we shall not distinguish between $\langle T^3\rangle$ and $\langle T\rangle^3$, $\langle \rho/T^3\rangle$ and $\langle\rho\rangle/\langle T\rangle^3$, etc., moreover, the characteristic values will be seen below to differ only slightly from the values at the core edge so that substituting Eq. (28) in Eq. (25) and rearranging, we obtain

$$L = 16\pi\times 10^{-\mu/2}\times\left(\frac{\sigma}{3(\nu+1)}\right)^{1/2}\left(\frac{XX_{\mathrm{CN}}}{\kappa}\right)^{1/2}$$
$$\times r_c^2\,({}^{\mathrm{H}}\rho_c)^{1/2} T_c^{(\nu+4)/2}. \tag{29}$$

Substituting the core edge radius and its density and temperature as estimated above, one finds immediately

$L \approx 4\times 10^{29}$ W.

Let us now go back and verify that the simplifications and substitutions are at least all internally consistent.

Using Eq. (28), the thermal gradient m works out to about 40 deg/m. Therefore, putting u equal to 0.85, the thickness $h_{\max}$ of the shell that is responsible for over 90% of the luminosity is about $0.15\times 3.8\times 10^7/40$, i.e., a little over 10^5 m. Now we have already seen that the density scale height is somewhat larger than this: Over the thickness of the shell, the density change is sufficiently small as to be neglected [especially as only its square root appears in Eq. (29)] and we may use the core edge value to calculate the luminosity. Clearly, r is essentially constant over the hydrogen-burning shell.

Our estimated value for the luminosity can be compared to the 9×10^{28} W given by Schwarzschild.[2]

V. THE STELLAR ENVELOPE

The luminosity of the hydrogen-burning shell is considerably larger than that of the star before core contraction began, $\sim 9\times 10^{26}$ W is the value obtained using the calculations in Celnikier[2]; the extra radiation is generated just outside the hydrogen-exhausted core in a very thin shell. This increase in the luminosity is a direct consequence of the evolution of the core and its associated rise in temperature and in reality takes a rather finite length of time.

How have the remaining $\frac{2}{3}$ of the stellar mass adjusted to the increase in luminosity.

Radiation can reach the outside of the star by one of two mechanisms—scattering from electrons and/or ions in the gas, or by inducing convection.

A. Radiation

In a normal star, photons leave via scattering processes that may be likened to a random walk so that the typical distance traversed by a photon before reaching the surface is of order $(R/\lambda)^2\lambda$, where R is the stellar radius and λ is the photon mean free path given by

$$\lambda = 1/\kappa\rho,$$

κ being the opacity as described earlier.

For the Sun, $\langle\kappa\rangle \approx 0.5$ and $\langle\rho\rangle \approx 10^3$, so that the time to "empty" the Sun of its radiation field is something like $(R/\lambda)^2(\lambda/c) \approx 10^{13}$ s. Now the total radiative energy is related to the mean internal temperature that is itself related to the mass via the requirement that there be hydrostatic equilibrium between gravity and pressure (see any elementary text); one obtains at once that the power radiated L is given by

$$L \propto M^3/\kappa, \tag{30}$$

the constant being a function of the chemical composition of the star and of the number of free particles per proton mass.

Of course, this can also be obtained from the standard expression for the radiative temperature gradient necessary to "drive" a certain power across a stellar shell, Eq. (26), by replacing the derivative by the ratio of the "typical" internal values, and the other quantities by their "typical" values, as in Eq. (28).

One can immediately deduce from Eq. (30) that a star of given mass, composition, and ionization state cannot evacuate energy at an arbitrary rate via the transfer of photons; the luminosity cannot rise by orders of magnitude unless the object becomes so extended and therefore so cold that the internal opacity, which depends critically on the presence of ions, drops very sharply.

This can be seen a little more explicitly by dividing Eq. (5) by Eq. (26), to give

$$\frac{dP}{dT}=\frac{64\pi GM\sigma}{3L\kappa}T^3. \quad (31)$$

In the vicinity of the core edge, the mass is essentially constant, equal to M_c (we shall see below just how far this approximate constancy extends), and so while a significant contribution to κ comes from electrons, this equation may be integrated to give

$$P=(64\pi GM_c\sigma/12L\kappa)\,T^4+\text{const.} \quad (32)$$

Now, we know that the pressure and temperature towards the edge of the star tend to zero; to a very rough approximation we may define the stellar edge as the point where $P=T=0$, allowing one (to the precision of this analysis) to neglect the integration constant so long as we restrict ourselves to the stellar interior—this is in any case implied by the condition $M\approx M_c$.

With this approximate relation for the variation of the pressure in the envelope, we obtain immediately from the ideal gas equation,

$$\rho=(64\pi GM_c\sigma m_p/12\alpha L\kappa k)\,T^3. \quad (33)$$

dP/dr can be written as $(dP/dT)(dT/dr)$; substituting this in Eq. (5) and using the above expressions for dP/dT and ρ, one obtains

$$\frac{dT}{dr}=-\frac{GM_c m_p}{4\alpha k r^2},$$

whence

$$T=(GM_c m_p/4\alpha\kappa)\,(1/r)+\text{const.} \quad (34)$$

Once again, since the temperature tends to zero towards the stellar edge, the integration constant may be neglected in approximate considerations of the stellar interior.

Equations (33) and (34) give an idea of how conditions in the envelope surrounding the active shell will vary as a function of distance; the essential feature of Eq. (33), the fact that the density is proportional to the third power of the temperature, can be shown (see, for example, Ref. 11) to result from some fairly general assumptions concerning the internal structure of any ordinary star.

These equations merit a little reflection.

The first reflection is technical. Equation (34) can in principle be used to estimate the temperature at the core edge required to support the envelope: One obtains 3.9×10^7 K.

For the second reflection, we note that the envelope density is inversely proportional to the luminosity. Let us use relations (34) and (33) to assess the temperature and density at, say, a solar radius from the center: One finds, respectively, $\sim10^6$ K and ~1 kg/m^3, which may be compared to Schwarzschild's[2] full numerical integration, 2×10^6 and 7, respectively. With these values one can verify that κ must still be virtually independent of temperature.

Next, substituting the density variation in Eq. (6) and integrating, one finds the envelope mass M_e beyond the core edge out to a radius r:

$$M_e=4\pi\left(\frac{GM_cM_p}{4\alpha k}\right)^4\frac{64\pi\sigma}{3\kappa L}\ln\frac{r}{r_c}$$
$$\approx 4\times10^{28}\ln(r/r_c). \quad (35)$$

At a solar radius from the center, this evaluates to $\sim10^{29}$ kg, i.e., a small fraction of the core mass.

The fall in density and temperature is entirely consistent with the way we imposed the boundary conditions; the envelope mass rises very slowly and so justifies (*a posteriori*!) the use of a constant central mass.

These estimates merely confirm, of course, the general conclusion: Radiation transfer cannot easily cope with the increased flux coming from the core unless the envelope density is lowered considerably. Indeed, if one were to fling aside any residual pretense of mathematical rigor and "forget" provisionally that Eq. (35) is a poor approximation when M_e becomes a finite fraction of the stellar mass, one could conclude that the envelope would have to extend to at least e^{100} times the core radius to enclose the original mass, i.e., the envelope has to be huge (this is qualitatively the essential conclusion of Applegate's[13] numerical integration; of course, the numbers themselves are somewhat different, as is only right and proper for a computed result). Actually, in practice, Eq. (35) is not as wildly wrong as one might think even when applied to an entire star: In the central hydrogen-burning phase when the luminosity is about 9×10^{26} W, the relation predicts a stellar radius of a little under 15 times the core radius, which is certainly too large...but not ludicrously so.

Inasmuch as the high luminosity is being invoked to drive the stellar envelope outward during this stage of the star's existence, one might think that the shell itself could be driven out, thereby lowering its average temperature and bringing its power production in line with the power evacuation, obviating the need for a huge envelope. To understand at least schematically (schematically since our calculations do not have the necessary precision) why this is not likely to be a feasible solution, let us recall the essential elements of Sec. IV A: The inner edge of the shell is at a temperature fixed by the compression of the core, and most of the luminosity is generated over a shell whose thickness is rather less than the local density scale height. Therefore, the temperature of the inner edge is virtually decoupled from the conditions in the envelope; the conditions in that part of the shell where most of the power is being produced will be driven rather by the core than by the envelope. This is the physical significance of the final result for the luminosity, in which only core edge parameters appear—since in this model the inner core is not a power source, the usual thermostatic control that a star has over its power production is no longer efficient.

A very large envelope suggests a relatively low surface temperature.

B. Convective energy transfer

Under certain conditions, energy can be transferred across a star by what is often loosely referred to as "convection": "blobs" of gas are set into motion through the agency of a suitable temperature gradient, maintain their identity for a certain distance l, called the mixing length, after which they "merge" with the background gas thereby transferring to it their higher energy. It is shown in standard texts that for a medium to be unstable with respect to this process, the temperature gradient must exceed a certain minimum fraction of the pressure gradient,

$$\frac{dT}{dr}=\frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dr}, \quad (36)$$

where γ is the ratio of specific heats, equal to $\frac{5}{3}$ for highly ionized stellar material, and P is the pressure.

One can obtain a convenient criterion for the onset of convection by substituting Eqs. (26) and (5) into Eq. (36):

$$\frac{3\alpha k}{64\pi\sigma m_p GM}\frac{\rho\kappa L}{T^3} > \frac{\gamma - 1}{\gamma} = 0.4. \quad (37)$$

The term convection is somewhat unfortunate, since it evokes an orderly cyclic fluid movement. This is certainly most unlikely to be the case within a star, where conditions are ripe for turbulence since the Rayleigh number is far beyond the value generally associated with the onset of turbulence (see, for example, Chiu[15]); it is better to visualize the "blobs" transferring their energy by means of a random walk entirely analogous to the random walk of photons, the mean free path being the "mixing length" l. Using this notion, we can assess the power that convection can transport across a star:

total "transfer time" $\approx (R/l)^2 l/v$,

where v is the typical velocity of a "blob";

energy available for transfer by convection

$$\approx (k/m_p)\Delta T\langle\rho\rangle(4\pi R^3/3),$$

where ΔT is the mean temperature difference between a blob and its environment;

power that can be transferred

$$\approx \frac{k}{m_p}\Delta T\langle\rho\rangle\frac{4\pi R^3}{3}\frac{vl}{R^2}$$

$$= \frac{k}{m_p}\Delta T\langle\rho\rangle\frac{4\pi R}{3}vl. \quad (38)$$

Note that when $l \ll R$, this is considerably smaller than the power that can be transferred by an orderly motion, $k\langle\rho\rangle\Delta T v 4\pi R^2/m_p$.

What can be put for ΔT, v, and l?

It is known that under stellar conditions convection starts as soon as criterion (37) is satisfied: This suggests that the "excess" temperature of a blob is very small. An upper limit of $\Delta T \approx 1$ deg (strange as this may seem to the uninitiated!) is often taken, as a round number. In experimentally controllable situations, the larger the container in which the convecting fluid is maintained, the smaller the value of ΔT; in those astronomical cases where an independent check can be carried out, such as in the outer layers of the Sun, via a comparison of observation with models, one finds that $\Delta T \ll 1$.

The convection velocity v is a rather arbitrary parameter. Schwarzschild[2] shows that its value might be as low as a few tens of meters per second; presumably it is unlikely to exceed the local velocity of sound, which in a gas whose internal temperature runs to millions of degrees, is of the order of 10^5 m/s.

The mixing length l is a no less tricky proposition, since in fact one does not even now possess a satisfactory theory of convection and turbulent energy transfer. In a gravitationally bound system, it is plausible for l to be related to the scale height at the point where the convection is driven, since the scale height is a measure of the region over which conditions start to change.

Can convection transport the extra energy across the star?

We note first that in the deep interior of the Sun, criterion (37) is not satisfied: Convection is not an important characteristic of the Sun, operating only near the surface where the temperature has dropped to below about 10^6 deg. This is a general feature of very highly ionized stellar material for which γ is always $\frac{5}{3}$, and for which the left-hand side of Eq. (37) is very close to 0.25 [indeed, exactly equal to 0.25 in the case of the idealized envelope discussed in Sec. V A, as one can show by substituting Eq. (33) in Eq. (37)]; in the Sun, convection sets in when the conditions allow partial recombination of the hydrogen and helium ions, thereby significantly lowering the value of γ.

Convection might conceivably be excited near to the outer edge of the hydrogen-burning shell, where the luminosity is changing rapidly and the strict envelope solution is presumably not valid. In this region, the scale height is only $\approx 10^6$ m; assuming sonic convection with $\Delta T \approx 1$, and assuming that the whole envelope is unstable to convection, the convective power that could possibly be transported through the star [Eq. (38)] turns out to be a little over 10^{27} W.

We note first that even this rather optimistic estimate is rather short of the power that must be evacuated. The energy that is not transported can only go to increase the internal energy of the star itself—the star must expand and cool.

Of still greater importance is the fact that the envelope itself is perfectly stable against convection, at least in those regions where hydrogen and helium are completely ionized.

One would expect convection to operate successfully in regions where the temperature has become sufficiently low; using the Sun as a guide, and using also the theoretical calculations for the onset of convection in a recombining medium given in Schwarzschild,[2] we conclude that the temperature must drop to below 10^6 deg, which in the case of the model analyzed in Sec. V A occurs at a radius that encloses only a very small fraction of the final mass.

Therefore, even if convection does start somewhere in the stellar envelope, it will not necessarily ensure that the overall radius will be small—it will merely be much smaller than it would otherwise have had to be.

VI. SOME FINAL COMMENTS

At some stage in its life, a star will have exhausted its central stock of hydrogen. It may come to the end of the stock relatively suddenly or relatively slowly, depending on the stellar mass; the details and the time scales of the subsequent evolution will depend on how the hydrogen is depleted. However, at some stage the star will find itself with a "dormant," compact, and ever-growing helium-rich core, as yet too cold to initiate helium fusion, but so hot (and still getting hotter) that a layer of hydrogen around its edge can continue to fuse via the extremely temperature-sensitive CN reaction. The power produced by this reaction exceeds by a very large factor the original stellar luminosity at the end of the main sequence, and this power can be evacuated only if the material surrounding the core becomes very tenuous and/or if convection can be initiated. This is the physical meaning of the full numerical computations and I have shown in this article how to estimate the luminosity of a star with a compact helium-rich core surrounded by a hydrogen-burning shell, using elementary techniques and straightforward physics. The high luminosity implies an inflated envelope since the temperature of the shell is very insensitive to the conditions in the external parts of the star: The usual "thermostatic control" that allows an ordinary star like the Sun to balance its energy production with its evacuation capability is in a sense inverted and the conditions in the core now drive the outer parts of the star. From this emerges also the idea that convection acts as a "safety valve" that helps to prevent the star from expanding too far even when it loses thermostatic control over its central regions. However, the safety valve is not completely effective, and the only real solution to the star's problem is the stabilization of the core through the appearance of a new thermonuclear reaction, or the emergence of a high degree of degeneracy.

A substantially similar discussion has been presented by Clayton[4] and also by Reddish,[11] albeit in a qualitative form.

One of the more distressing features of trying to understand the nature of red giant structure as it is presented in the literature is the contrast between the affirmative statements made and the arbitrariness of many of the parameters that are actually used in the numerical analyses. Convection is clearly an essential prerequisite if a star in this phase of its existence is to survive; while the account given above is without doubt somewhat caricatural and truncated, the essential physical problem of knowing whether convection *can* act efficiently is no better treated in the detailed numerical models: The arbitrary parameters presented here are no less arbitrary in the numerical models—rather, they are adjusted so that the models give something that looks reasonable. This is not a criticism of stellar modeling, rather of the way the results are interpreted afterward.

The methods presented in this article do not, and indeed cannot, lead to the calculation of when the various "happenings" occur: The results should be taken in the spirit of a "snapshot" showing an idealized stellar state at a certain point in its evolution. By what detailed path it reached that state, and after what time, is quite another story; note, too, that the actual mechanism that initiates and maintains the expansion is not identified in this article—only what the final state should be like.

ACKNOWLEDGMENTS

I am grateful to N. Meyer, who highlighted a number of ambiguities in this article and suggested many improvements, and I would like to express my gratitude to an anonymous referee whose clear competence in the matter of stellar structure helped bring my ideas into better focus.

[1] D. A. Leahy, "A demonstration calculation of self-consistent stellar models with convective cores," Am. J. Phys. **56**, 68–72 (1988).

[2] Martin Schwarzschild, *Structure and Evolution of the Stars* (Princeton U.P., Princeton, NJ, 1958; Dover, New York, 1965).

[3] Edwin E. Salpeter, "Dimensional ratios and stellar structure," in *Perspectives in Modern Physics* (Wiley, New York, 1966), pp. 463–475.

[4] Donald D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis* (McGraw-Hill, New York, 1968).

[5] Victor Weisskopf, "Of atoms, mountains and stars: A study in qualitative physics," Science **187**, 605–612 (1975).

[6] Howard S. Goldberg and Michael D. Scadron, *Physics of Stellar Evolution and Cosmology* (Gordon and Breach, New York, 1981).

[7] George Greenstein, "Order of magnitude 'theory' of stellar structure," Am. J. Phys. **55**, 804–810 (1987).

[8] Ludwik M. Celnikier, "Stellar structure and the art of building boats," Am. J. Phys. **57**, 159–166 (1989).

[9] Ludwik M. Celnikier, "Another simple model for energy emission by black holes," Am. J. Phys. **48**, 725–727 (1980).

[10] Icko Iben, "Stellar evolution within and off the main sequence," Ann. Rev. Astron. Astrophys. **5**, 571–626 (1967).

[11] V. C. Reddish, *The Physics of Stellar Interiors* (Edinburgh U.P., Edinburgh, 1974), pp. 50–69.

[12] Amos Yahil and Leo Van den Horn, "Why do giants puff up?" Astrophys. J. **296**, 554–564 (1985).

[13] James H. Applegate, "Why do stars become red giants?" Astrophys. **329**, 803–807 (1988).

[14] Roger J. Tayler, "What makes red giants?" Nature **335**, 14–15 (1988).

[15] Hong-Yee Chiu, *Stellar Physics* (Blaisdell, Reading, MA, 1968), Vol. 1, pp. 70–71.

[16] William K. Rose, *Astrophysics* (Holt, Rinehart and Winston, New York, 1973), pp. 127–128.

[17] John R. Acton and Patrick T. Squire, *Solving Equations with Physical Understanding* (Hilger, Bristol, England, 1985).

Reconciliation of esu and mksa units in nonlinear optics

I. M. Skinner and S. J. Garth

Department of Mathematics, University College (UNSW), Australian Defence Force Academy, Canberra 2600, Australia

(Received 8 July 1988; accepted for publication 29 March 1989)

An explanation is provided of the differences between the mksa and esu systems used in the study of nonlinearities in the optical properties of materials used in optical waveguides. Means of converting both units and quantities between the two are presented, and the relationship between the electric susceptibility tensors defined in the two systems is demonstrated.

I. INTRODUCTION

Because optical waveguides confine light to small regions, it requires only moderate levels of illuminating power for the guide to become nonlinear in its behavior. Many effects, such as Raman and Brillouin scattering, self-focusing, and frequency generation, have been observed and analyzed.[1] This branch of science is currently attracting great interest,[2] but, to the new student or researcher, one early difficulty is the nature of the units used to describe phenomena. Historically, electromagnetic theory has used Gaussian units, of which the esu (electrostatic units) sys-