Reconstructing images in astrophysics

Part 1: Claude Aime
Part 2: Céline Theys

Université de Nice Sophia Antipolis
UMR 7293 J.L. Lagrange
Observatoire de la Côte d'Azur
Parc Valrose 06108 Nice Cedex

Besançon 2014

Outline part 1

Physical basis of optical imaging

- The Huygens - Fresnel model
- Point Spread Function and Optical transfer function
- Examples on a test object

A preliminary introduction to the inverse imaging problem

- The inverse filter: a band limited constraint
- Non-linear approches: impainting the Fourier space

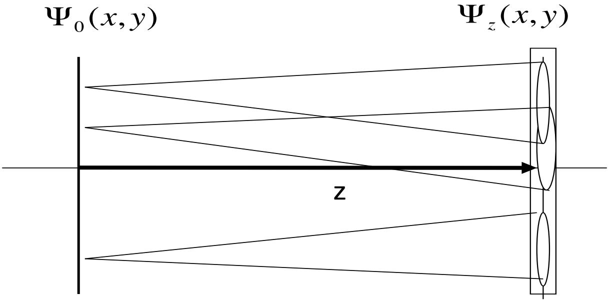
Image formation with a telescope

The semi-classical theory of image formation and photodetection

- **A coherent process**:
  The PSF is obtained using Fourier optics.
- **An incoherent addition in intensity**:
  The object image relationship is a convolution (or a Fredholm integral for space-variant response).
- **Photodetection**:
  A Poisson process $p(n/m)$ describes the observed number of photoelectrons n from the mean expected value m).

For a basic course in optics with illustrations (by Eric Aristidi, in French):
http://www.unice.fr/DeptPhys/optique/optique.html

Free space propagation: the Huygens-Fresnel model



$$\underbrace{\Psi_z(x,y)}_{wave\ at\ z} = \underbrace{\Psi_0(x,y)}_{wave\ at\ z=0} * \underbrace{\frac{1}{i\lambda z}\exp(i\pi\frac{x^2+y^2}{\lambda z})}_{Huygens-Fresnel\ wavelet} \quad (1)$$

The Fresnel integral

With $r = (x, y)$, expanding the 2D convolution, we have:

$$\Psi_z(r) = \frac{1}{i\lambda z} \exp(i\pi \frac{r^2}{\lambda z}) \int [\Psi_0(\xi) \exp(i\pi \frac{\xi^2}{\lambda z})] \exp(-2i\pi \xi \frac{r}{\lambda z}) d\xi \qquad (2)$$

or

$$\Psi_z(r) = \frac{1}{i\lambda z} \exp(i\pi \frac{r^2}{\lambda z}) \Im_{\frac{r}{\lambda z}} [\Psi_0(r) \exp(i\pi \frac{r^2}{\lambda z})] \qquad (3)$$

where $\Im_{\frac{r}{\lambda z}}$ means the Fourier transform for the "frequency" $r/\lambda z$.

Lenses as quadratic phase terms.

Converging lens

$$\mathcal{L}_{+F}(x,y) = e^{-i\pi\frac{x^2+y^2}{\lambda F}}$$

Diverging lens

$$\mathcal{L}_{-F}(x,y) = e^{+i\pi\frac{x^2+y^2}{\lambda F}}$$

These expressions can be obtained very simply.

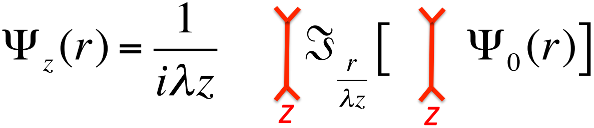
Note that $\mathcal{L}_{-F}(x,y) \times \mathcal{L}_{+F}(x,y) = 1$

Back to the Fresnel diffraction

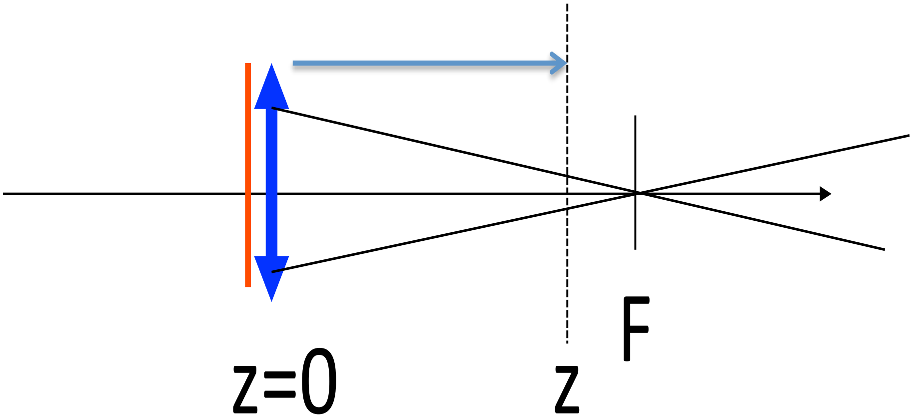
We can write:

$$\Psi_z(r) = \frac{1}{i\lambda z}\,\mathrm{exp}(i\pi\frac{r^2}{\lambda z})\Im_{\frac{r}{\lambda z}}[\Psi_0(r)\,\mathrm{exp}(i\pi\frac{r^2}{\lambda z})] \tag{4}$$

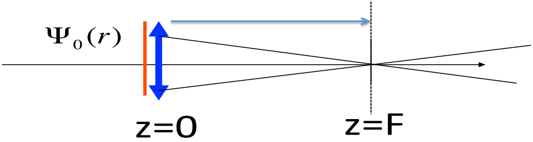
as:

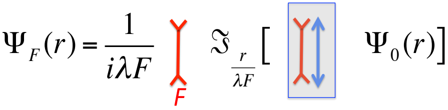


Propagation of a wave after a converging lens



In the focal plane at z=F





Converging and diverging lenses cancel, and we have simply **the Fourier transform of the wave on the lens** (to a quadratic phase term):

$$\Psi_F(r) = \frac{1}{i\lambda F} \exp(i\pi \frac{r^2}{\lambda F}) \hat{\Psi}_0(\frac{r}{\lambda F}) \quad (5)$$

Point Spread Function

For a wave $\Psi_0(r)$ arriving on the aperture $P_0(r)$, we denotes $P(r)$:

$$P(r) = \Psi_0(r) P_0(r), \tag{6}$$

in the focal plane:

$$|\Psi_F(r)|^2 = \frac{1}{\lambda^2 F^2} |\hat{P}(\frac{r}{\lambda F})|^2 \tag{7}$$

It is convenient to use angular units $\alpha = (\alpha_x, \alpha_y)$, and define the PSF:

$$H(\alpha) = \frac{1}{\lambda^2 \mathbb{S}} |\hat{P}(\frac{\alpha}{\lambda})|^2 \tag{8}$$

where $\mathbb{S}$ is the telescope area, and H is normalized so that $H(\alpha) * 1 = 1$.

Object - image relationship: the Fredholm integral

For a point source in the angular direction β (intensity $O(\beta)$), the elementary observed focal plane intensity is:

$$dI(\alpha) = O(\beta)H(\alpha, \beta) \tag{9}$$

Summing for all β directions, the intensity in the focal plane becomes:

$$I(\alpha) = \int O(\beta)H(\alpha, \beta)d\beta \tag{10}$$

which is a Fredholm integral.

The convolution relationship

Assuming a space-invariant response, we have

$$I(\alpha) = \int O(\beta)H(\alpha + \beta)d\beta \tag{11}$$

and there is a simple relation of convolution between the image, object and the PSF.

$$I(\alpha) = O(\alpha) * H(-\alpha) \tag{12}$$

where we recall that the convolution is 2D. The minus sign comes from the inversion of the geometrical image by the lens.

The optical transfer function T(u)

In the Fourier space, with $u = (u_x, u_y)$

$$\hat{I}(u) = \hat{O}(u).T(u) \tag{13}$$

where

$$T(u) = \frac{1}{\mathbb{S}} \int P(r)P^*(r - \lambda u)dr \tag{14}$$

is a low pass 2D filter, or a band-pass filter for diluted apertures.

PSF and OTF: perfect circular aperture.

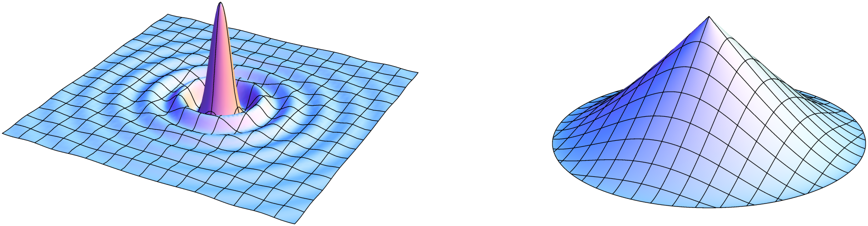
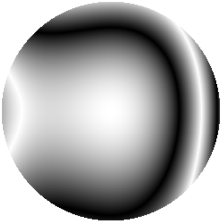
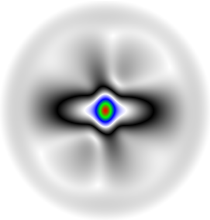
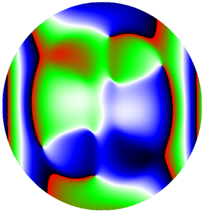


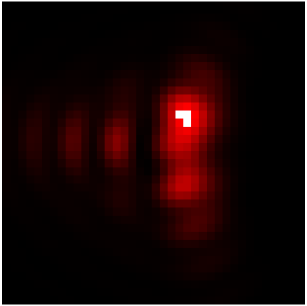
Figure: 3D representation of the Airy function and corresponding OTF. Cut-off frequency: $|u_c| = D/\lambda$

Aberrated circular aperture, OTF and PSF

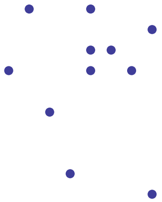


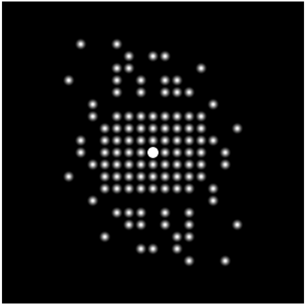


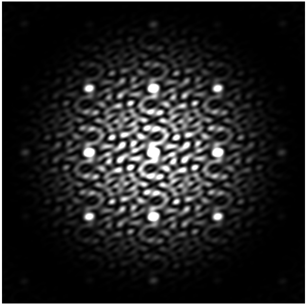




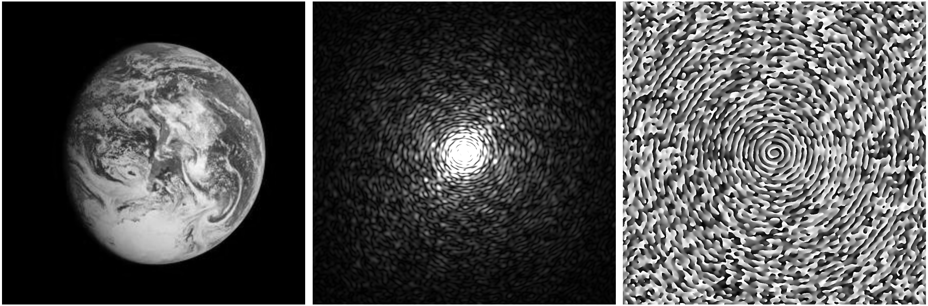
Diluted apertures, OTF and PSF



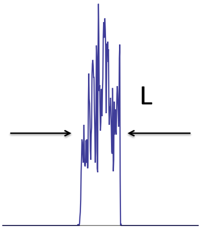




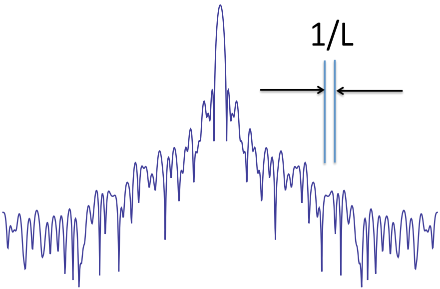
Example of an object and its Fourier transform: log scale modulus and phase



Characteristic sizes: cuts of the object and modulus of its Fourier transform



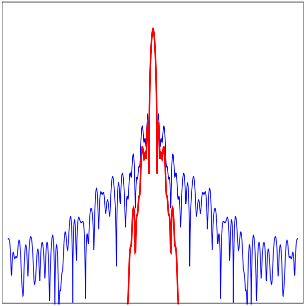
Object



Modulus of the Fourier transform

Blurred image

Low pass filter effect of the telescope MTF



The inverse filter: a linear approach

The inverse filter

From:

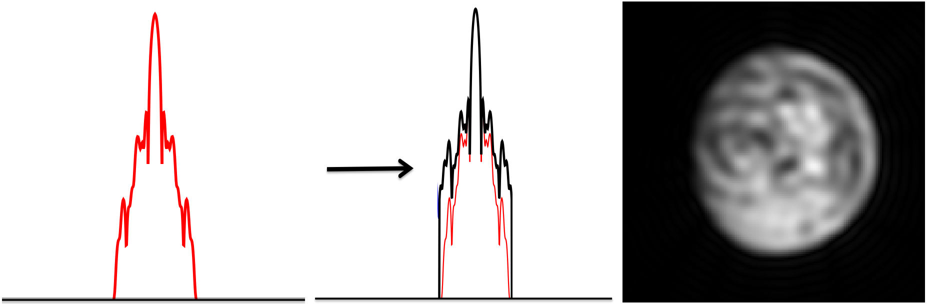
$$\hat{I}(u) = \hat{O}(u).T(u) \tag{15}$$

One may consider the following simple procedure:

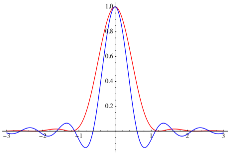
$$\begin{aligned} \hat{O}_{est}(u) &= \frac{\hat{I}(u)}{T(u)} \; \textit{for} \;\; T(u) \neq 0, \; \textit{and} \;\; = 0 \; \textit{otherwise} \\ O_{est}(r) &= \Im^{-1}[\hat{O}_{est}(u)] \end{aligned} \tag{16}$$

As a result, $\hat{O}_{est}(u)$ is non-zero only where $T(u) \neq 0$.
Moreover $\hat{I}(u)$ may be contaminated by noises: photon noise, additive noise ...which will be amplified by the division....

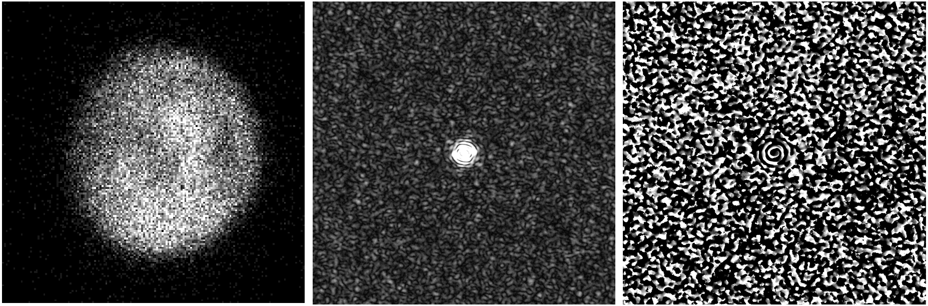
Inverse filter on a noiseless image: Gibbs effects



The reconstructed PSF has negative parts



Example of an image with 5×10^4 photons

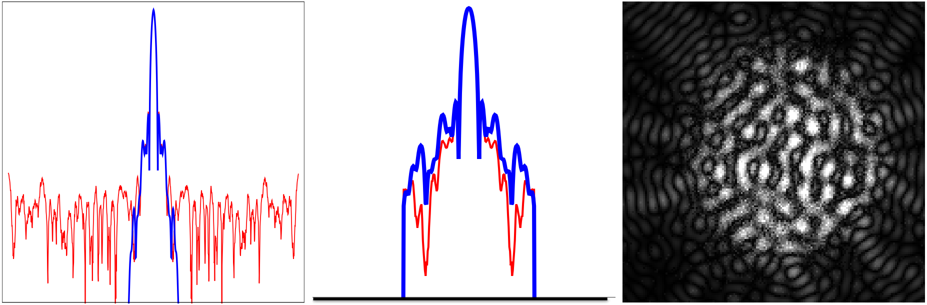


Photodetected image, modulus and phase of the spectrum.

The number of photons in a pixel is obtained using the Poisson law:

$$p(n/m) = e^{-m}\frac{m^n}{n!}$$

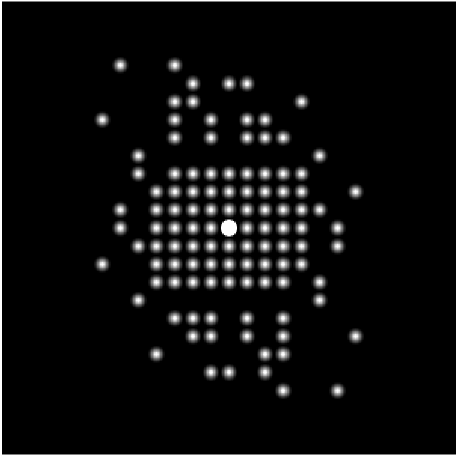
Inverse filter applied to the noisy image



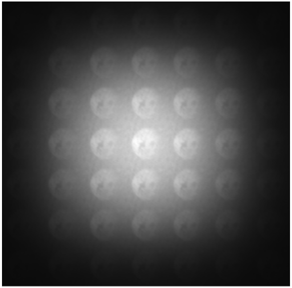
The result may be somewhat improved using optimal Wiener filter....
See Brault, J. W., & White, O. R. 1971, A&A, 13, 169, for example

Non-linear approaches

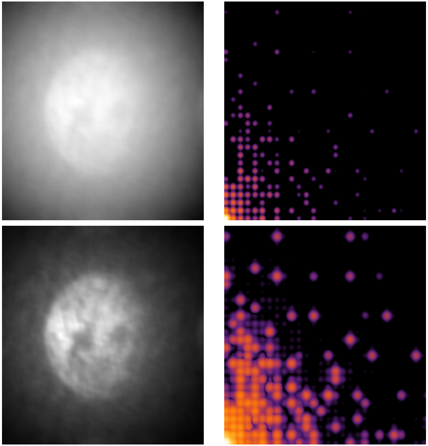
On the example of diluted apertures (example of OTF)



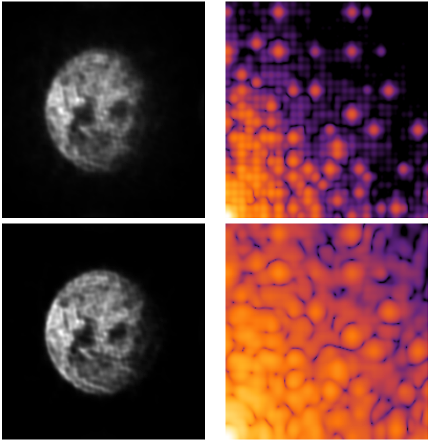
Focal plane image with replica



In this example, the non-linear approach is Richardson-Lucy, an iterative algorithm described in sect. 2



Increasing the iteration number



A non-linear approach may impaint the Fourier space i.e. "invent" missing angular frequencies