Image reconstruction

Part II
Inverse problems - A methodology

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Outline

A simple example: optical images deconvolution

Mathematical Model

$$y(n, m) = \sum_{k,l} h(k, l)x(n - k, m - l) + (b(n, m)) \qquad (1)$$

The impulse response h includes:

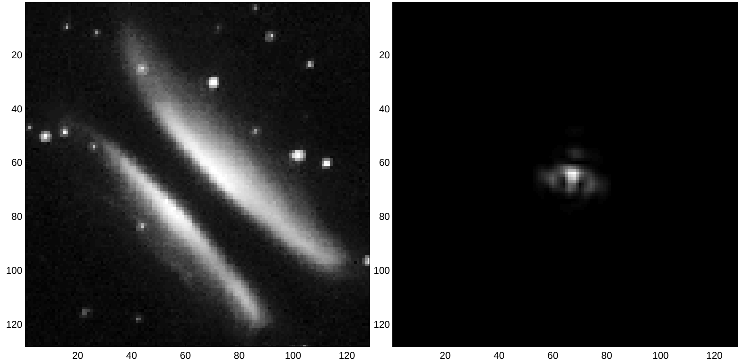
- The limited aperture of the optical system.
- A move of the object during the exposition.
- Atmospheric turbulences.
- All of these phenomena (astrophysical: 1 + 3).

b: measure and model errors

x: object

y: image

HST data, Psf


20
40
60
80
100
120
20 40 60 80 100 120
20
40
60
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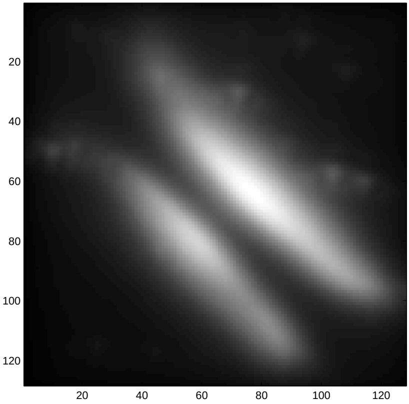
120
20 40 60 80 100 120

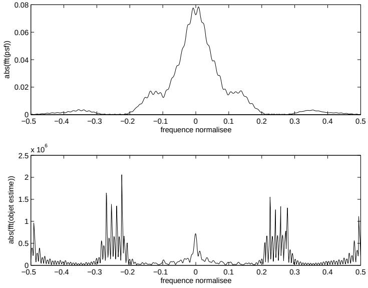
Result of the convolution



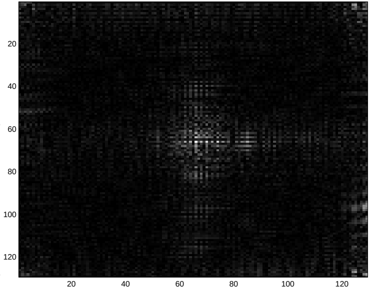
Solution in the Fourier plan

$$Y(f_x, f_y) = H(f_x, f_y)X(f_x, f_y) \quad (2)$$

$|H(f_x = 0, f_y)|$, $|X(f_x = 0, f_y)|$



Estimated Object



Remarks

$$X(f_x, f_y) = \frac{Y(f_x, f_y)}{H(f_x, f_y)} - \left(\frac{B(f_x, f_y)}{H(f_x, f_y)}\right) \quad (3)$$

- Lowpass system with transmission zeros.
- Without noise, amplification of the components at frequencies corresponding to the transmission zeros.
- With noise, amplification of the noise at the same frequencies.
- *Signal processing: if the noise is Gaussian, it is "the best" solution (unbiased and minimum variance estimator)*

Conclusion: ill posed problem!!

Ill posed problems

$$y = S[x] \qquad (4)$$

y: measures; x: object; $S[]$: known transformation.
Problem: recover x from y?

Well posed problems: Hadamard conditions

1. Existency. The inverse operator exists, there is at least one solution x.
2. Unicity. The solution x is unique.
3. Stability. The solution is stable.

First kind Fredholm equation

Monodimensional case ($k(s,r)$: kernel, $\mathbf{x}(\mathbf{r})$: unknown)

$$y(s) = \int_a^b k(s,r)x(r)dr \tag{5}$$

For translation invariant kernel, **convolution equation**:

$$y(s) = \int_{-\infty}^{\infty} k(s-r)x(r)dr \tag{6}$$

First kind Fredholm equation

Bidimensional case (optical imagery)

$$y(s,t) = \int_a^b \int_c^d k(s,t,r,v)x(r,v)drdv \tag{7}$$

For translation invariant kernel, **bi-dimensional convolution equation**:

$$y(s,t) = \int_a^b \int_c^d k(s-r,t-v)x(r,v)drdv \tag{8}$$

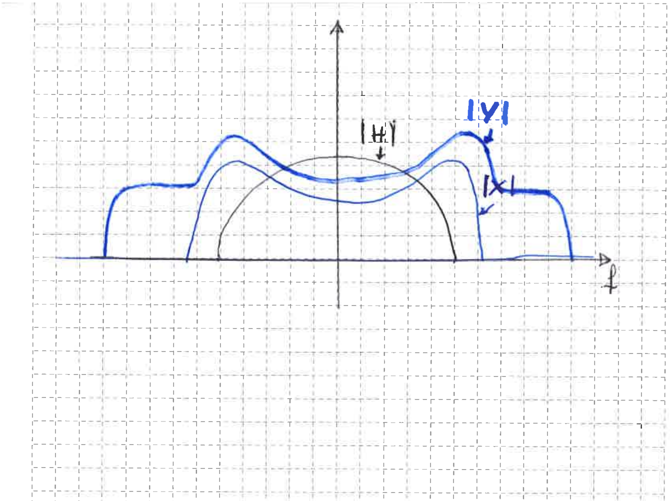
And separable:

$$y(s,t) = \int_a^b \int_c^d l(s-r)m(t-v)x(r,v)drdv \tag{9}$$

Existency

Existency

"If an unlimited bandwidth error (white noise for example) or outside the bandwidth is added on y then there is no solution



Unicity

Riemann-Lebesgue theorem

$$\lim_{\alpha \to \infty} \int_a^b k(s,r)(x(r) + \sin(\alpha r))dr = \int_a^b k(s,r)x(r)dr \qquad (10)$$

"If high component frequencies are added to x, they are removed by the transformation, i.e x is recovered up to high frequencies"

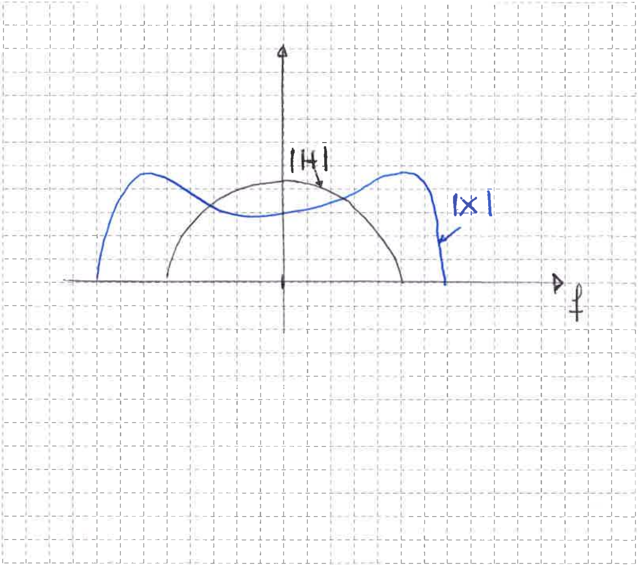
More generally $x(r) = u(r) + v(r)$

$$\int_a^b k(s,r)(u(r) + v(r))dr = \int_a^b k(s,r)u(r)dr \qquad (11)$$

with $v(r)$ orthogonal to $k(r,s)$.

"x is recovered up to component frequencies outside the bandwidth of the instrument"

Unicity



Discretization

x and y belong to finite dimensional spaces and k is then a matrix H:

$$\mathbf{y} = H\mathbf{x} + (\mathbf{b}) \tag{12}$$

Structure of H - One dimensional convolution

$$y_n = \sum_{i=-Q}^{P} x_{n-i} h_i \tag{13}$$

One value of $y \rightarrow P + Q + 1$ values of x
Shift of one for each new value of y
N values of $y \rightarrow P + Q + N$ values of x

Structure of H - One dimensional convolution

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} h_P & \dots & h_0 & \dots & h_{-Q} & 0 & \dots \\ 0 & h_P & \dots & h_0 & \dots & h_{-Q} & 0 \\ \vdots & \ddots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & h_P & \dots & h_0 & \dots & h_{-Q} \end{pmatrix} \begin{pmatrix} x_{-P+1} \\ \vdots \\ x_0 \\ x_1 \\ \vdots \\ x_{1+Q} \\ \vdots \\ x_N \\ \vdots \\ x_{N+Q} \end{pmatrix}$$

Using the DFT

- Underdetermined system:
  $dim(\mathbf{x}) = N + P + Q > dim(\mathbf{y}) = N$.
- Add $P + Q$ rows to H to get a square circulant matrix
  $dim(H) = (N + P + Q) \times (N + P + Q)$.
- Then we can compute $\mathbf{y}$ by
  1. DFT of $\mathbf{x}$ (dimension L).
  2. DFT of the first row of H, (dimension L).
  3. Product term by term of the two DFTs.
  4. IDFT of the result.
  5. Extraction of the N "correct" values from the $N + P + Q$ values of $\mathbf{y}$.

Structure of H - Two dimensional convolution

$$y_{n,k} = \sum_{i=-Q_1}^{P_1} \sum_{j=-Q_2}^{P_2} h_{i,j} x_{(n-i,k-j)} \tag{14}$$

$$y_{1,1} = \sum_i \left(h_{(i,-Q_2)} x_{(1-i,1+Q_2)} + h_{(i,-Q_2+1)} x_{(1-i,Q_2)} + \ldots + h_{(i,P_2)} x_{(1-i,1-P_2)} \right) \tag{15}$$

One value of $y_{n,k} \rightarrow (P_1 + Q_1 + 1)(P_2 + Q_2 + 1)$ values of $x_{n,k}$
N^2 values $\rightarrow (P_1 + Q_1 + N^2)(P_2 + Q_2 + N^2)$ values of $x_{n,k}$

$$
\begin{pmatrix} y_{1,1} \\ y_{1,2} \\ \vdots \\ y_{N,N} \end{pmatrix} =
\begin{pmatrix}
h_{P_1,P_2} & \dots & h_{P_1,-Q_2} & \dots & h_{-Q_1,-Q_2} & 0 & \dots & \\
0 & h_{P_1,P_2} & \dots & h_{P_1,-Q_2} & \dots & h_{-Q_1,-Q_2} & \dots & \\
 & \ddots & & \ddots & & \ddots & & \ddots \\
0 & \dots & h_{P_1,P_2} & \dots & h_{P_1,-Q_2} & \dots & h_{-Q_1,-Q_2} &
\end{pmatrix}
\begin{pmatrix} x_{(1-P_1,1-P_2)} \\ \vdots \\ x_{(1+Q_1,1+Q_2)} \\ x_{(2-P_1,2-P_2)} \\ \vdots \\ x_{(2+Q_1,2+Q_2)} \\ \vdots \\ x_{(N+Q_1,N+Q_2)} \end{pmatrix} \quad (16)
$$

Solutions

Whatever dimensions and without noise or unknown noise

$$\mathbf{x}_{est} = arg \min_{\mathbf{x}} ||\mathbf{y} - H\mathbf{x}||^2 = (H^T H)^{-1} H^T \mathbf{y} \qquad (17)$$

With i.i.d noise

$$\mathbf{x}_{est} = (H^T H)^{-1} H^T \mathbf{y} \qquad (18)$$

With known autocorrelation matrix

$$\mathbf{x}_{est} = (H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{y} \qquad R = E[\mathbf{b}\mathbf{b}^T] \qquad (19)$$

$(H^T H)^{-1} H^T = H^\dagger$: generalized inverse matrix

Conclusion: bad conditioning

In the discretised case, there is always a solution, the problem is the inversion of H or $H^T H \rightarrow H$ **is bad conditioned**.

Condition number

$$c = \frac{\lambda_{max}}{\lambda_{min}} \tag{20}$$

λ_{max}: maximal eigenvalue, λ_{min}: minimal eigenvalue
well-conditioned: $c \approx 1$, bad-conditioned: $c \gg 1$,
(introducing example: $c = 4.10^5$)

Solutions to bad conditioning

Truncated eigendecomposition

- The eigendecomposition allows to increase the conditioning by suppressing the smaller eigenvalues.
- If the λ_i are arranged in descending order and the eigenvectors are arranged in the order of the eigenvalues, one can simply truncate the decomposition by stopping the summation to main eigenvalues
- $(H^T H)^{-1} = \sum_{i=1}^{\mathbf{K}} \frac{1}{\lambda_i^2} \mathbf{v}_i \mathbf{v}_i^T$

Problems

- Choice of the truncation?
- No physical information on the object ...

Adding a priori - Regularization

- By adding a priori, we give up the true solution, anyway the true solution is lost.
- We search a "stabilized" solution.
- Signal processing: biased estimator.

Minimization of a composite criterion

$$J(\mathbf{x}) = J_1(\mathbf{y}, \mathbf{x}) + \gamma J_2(\mathbf{x}) \tag{21}$$

J_1: fidelity term, J_2 **a priori term on x or regularization term**, γ: regularization coefficient.
Constraint: $J(\mathbf{x})$ must be convex!

Regularization

Smoothness constraint to reduce the amplification of the noise.

Tikhonov regularization

$$J_2(\mathbf{x}) = ||D\mathbf{x}||^2 \tag{22}$$

With D a linear operator.

- $J_2(\mathbf{x}) = ||\mathbf{x}||^2 = ||\mathbf{x} - 0||^2$ "we search to minimize the norm of the solution"

Regularization

- $J_2(\mathbf{x}) = ||C\mathbf{x}||^2$ with C first derivative operator "we search to minimize the norm of the derivative of the solution"

$$C = \begin{pmatrix} 1 & -1 & 0 & \dots & \dots \\ 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- $J_2(\mathbf{x}) = ||L\mathbf{x}||^2$ with L second derivative operator "we search to minimize the norm of the second derivative of the solution"

$$L = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Regularization

$$J_2(\mathbf{x}) = ||\mathbf{x} - \mathbf{x}_{priori}||^2{}_W = (\mathbf{x} - \mathbf{x}_{priori})^T W^{-1} (\mathbf{x} - \mathbf{x}_{priori}) \quad (23)$$

- This distance corresponds to a Gaussian a priori on $\mathbf{x}$ of mean $\mathbf{x}_{priori}$ and covariance matrix W.
- If the fidelity term is also quadratic, then:

$$\begin{aligned} J(\mathbf{x}) &= ||\mathbf{y} - H\mathbf{x}||^2{}_R + \gamma ||\mathbf{x} - \mathbf{x}_{priori}||^2{}_W \\ &= (\mathbf{y} - H\mathbf{x})^T R^{-1} (\mathbf{y} - H\mathbf{x}) + \gamma (\mathbf{x} - \mathbf{x}_{priori})^T W^{-1} (\mathbf{x} - \mathbf{x}_{priori}) \end{aligned}$$

Regularization

Explicit minimizer

$$\mathbf{x}_{est} = (H^T R^{-1} H + \gamma W^{-1})^{-1} (H^T R^{-1} \mathbf{y} + \gamma W^{-1} \mathbf{x}_{priori}) \quad (24)$$

For $\gamma = 0$, we find the generalized inverse solution
For $\gamma \to \infty$, $\hat{\mathbf{x}} \to \mathbf{x}_{priori}$

Remark: We add γ on the diagonal of $H^T H$ (if W is diagonal)

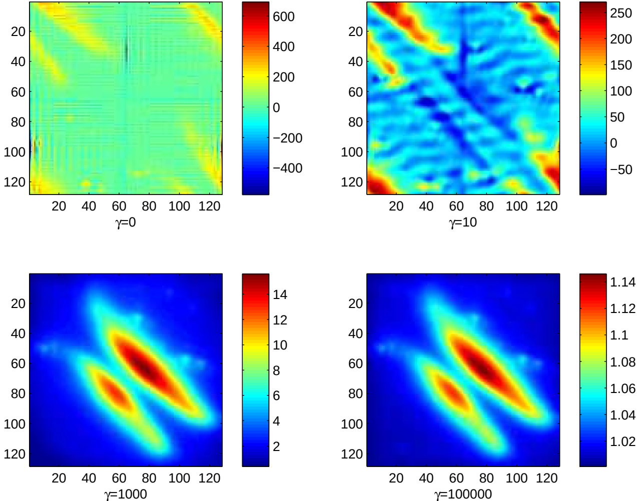
Simulation example

```
>>image=conv2(psf128,objet19,'same'); >>imagebruitee=image+randn(128); >>gamma=10

>>dirac=zeros(128); >>dirac(64,64)=1;

>>objetest=ifft2((fft2(imagebruitee)+gamma*fft2(ones(128)))

./(fft2(psf128)+gamma*abs(fft2(dirac)));
```



Regularization

How to choose γ?

- Find a value of the regularization coefficient such that:

$$\mathbf{x}_{est} = arg\min_{\mathbf{x}}(J_1(\mathbf{y}, \mathbf{x}) + \gamma J_2(\mathbf{x})) \tag{25}$$

be as close as possible to $\mathbf{x}$:

$$\gamma_{est} = arg\min_{\gamma}(||\mathbf{x}_{est} - \mathbf{x}||^2) \tag{26}$$

- No explicit solution in practice since $\mathbf{x}$ is not known.
- There are some results in the quadratic case: "curve in L", cross validation ...

Iterative methods?

- No explicit solution
- Explicit solution but high order inversion
- Inequalities constraints (positivity ...) $\rightarrow$ non linear equation

$$[\nabla J(x^\star)]x^\star = 0 \tag{27}$$

*: optimum

- Stop before convergence! (without regularization)

Descent algorithm

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha^{(k)}\mathbf{d}^{(k)} \tag{28}$$

$\mathbf{d}^{(k)}$: descent direction/ $J(\hat{\mathbf{x}}^{(k+1)}) \leq J(\hat{\mathbf{x}}^{(k)})$
$\alpha^{(k)}$: step size

Richardson Lucy algorithm

- Poisson distribution:

$$\mathbf{y} = \mathcal{P}(H\mathbf{x}) \tag{29}$$

- Fidelity term:

$$J(\mathbf{x}) = \sum_{i=1}^{N} ((H\mathbf{x})_i - y_i \log(H\mathbf{x})_i) \tag{30}$$

- Descent direction:

$$\mathbf{d}^{(k)} = -[\nabla_{\mathbf{x}} J(\mathbf{x})].\mathbf{x} = H^T(\mathbf{y}./H\mathbf{x} - 1_N).\mathbf{x} \quad \textit{and} \quad \alpha^{(k)} = \alpha = 1 \tag{31}$$

Richardson Lucy algorithm

$$x_i^{(k+1)} = x_i^{(k)} [H^T]_i \frac{y_i}{[H\mathbf{x}^{(k)}]_i} \tag{32}$$

Image Space Reconstruction algorithm

- Additive Gaussian noise:

$$\mathbf{y} = H\mathbf{x} + \mathbf{b} \qquad y_i = (H\mathbf{x})_i + b_i \qquad R = E(\mathbf{b}^T\mathbf{b}) \tag{33}$$

- Fidelity term:

$$J(\mathbf{x}) = ||\mathbf{y} - H\mathbf{x}||_R^2 = (\mathbf{y} - H\mathbf{x})^T R^{-1} (\mathbf{y} - H\mathbf{x}) \tag{34}$$

- Descent direction:

$$\mathbf{d}^{(k)} = -\frac{[\nabla_{\mathbf{x}} J(\mathbf{x})]}{H^T R H\mathbf{x}}.\mathbf{x} = \left(\frac{-H^T R H\mathbf{x} + H^T R\mathbf{y}}{H^T R H\mathbf{x}} \right).\mathbf{x} \tag{35}$$

For $\alpha^{(k)} = \alpha = 1$, ISRA algorithm

$$x_i^{(k+1)} = x_i^{(k)} \frac{[H^T R\mathbf{y}]_i}{[H^T R H\mathbf{x}^{(k)}]_i} \tag{36}$$

Conclusion

Observation by an optical system is an ill-posed problem

→ Direct inversion gives irrelevant result
→ **Regularization** = introduction of smoothness a priori
→ **Iterative methods** for explicit gradient, non linear constraints or/and high dimensional problem
→ **If no invariant psf, inversion of a very high dimensional matrix**