Asymptotic structure of the acoustic frequency spectrum I

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Outline

1. What motivates an asymptotic analysis of stellar pulsation modes ?
2. Regular frequency spectra
   - Simple examples
   - The non-rotating spherically symmetric star
   - The acoustic ray dynamics

What motivates an asymptotic analysis of stellar pulsation modes ?

The asymptotic analysis enables to:

- Classify the modes/ Understand the structure of the frequency spectrum (large and small separations)
- Guide the identification of the observed frequencies (echelle diagram)
- Construct tools for asteroseismology (inversion techniques, time-distance analysis)

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Objectives of this lecture

- Present the basic tools necessary to derive the asymptotic formulas in spherical stars
- Present an alternative method based on acoustic ray dynamics
- Show that non-regular spectra exist in more complex systems
- Present current research on the asymptotic theory in rotating stars

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- Regular frequency spectra
  - Simple examples
    - Vibrating string
    - Sound waves in a homogeneous sphere
  - Non-rotating spherically symmetric star
    - The WKB approximation
    - The turning points
    - The asymptotic formula
  - The acoustic ray dynamics
    - The WKB approximation
    - The Hamiltonian dynamics
    - The EBK semiclassical quantization

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Outline II

- Irregular spectra
  - The frequency statistics of classically chaotic system
  - Experimental signatures of quantum chaos
- Irregular/regular frequency spectra in mixed systems
  - The case of uniformly rotating polytropic stars

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The vibrating string I

- The (small) vertical displacement of the string, $U(x,t)$, verifies the 1D wave equation :

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

where $c = \sqrt{T/\mu}$ is a velocity formed form the string tension T and its linear density μ.
Time-harmonic solutions $U(x,t) = \Re[u(x)\exp(-i\omega t)]$ read:

$$\frac{d^2 u}{dx^2} + k^2 u = 0 \quad \Rightarrow \quad u(x) = A_0 \sin(kx+\phi) \quad \text{where} \quad k = \frac{\omega}{c}$$

- The string is fixed at both ends :

$$u(0) = 0 \quad \Rightarrow \quad \phi = 0$$

$$u(L) = 0 \quad \Rightarrow \quad kL = n\pi \quad \Rightarrow \quad \omega = n\pi\frac{c}{L}$$

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The vibrating string II

- A discrete spectrum : $\omega_n = n\pi \frac{c}{L}, \quad n \in \mathrm{N}$
- Frequency spacing : $\Delta_n = \omega_{n+1} - \omega_n = \pi \frac{c}{L}$

Governing equation

- 1D Helmholtz equation : $\frac{d^2u}{dx^2} + k^2 u = 0, \quad k$ constant

Spectrum properties

- Regular spectrum : $\omega = f(n)$
- Δ_n is constant

Adiabatic sound waves in a homogeneous sphere I

The Eulerian pressure perturbation, $P(\vec{x}, t)$, verifies the wave equation :

$$\Delta P - \frac{1}{c_s^2}\frac{\partial^2 P}{\partial t^2} = 0 \quad \text{where } \; c_s \; \text{ is the sound speed}$$

This can be simply derivated from the general equations governing adiabatic oscillations in a non-rotating star :

$$\partial_t \rho + \vec{\nabla} \cdot (\rho_0 \vec{u}) = 0,$$

$$\rho_0 \partial_t \vec{u} = -\vec{\nabla} P + \rho \vec{g}_0,$$

$$\partial_t P + \vec{u} \cdot \vec{\nabla} P_0 = c_s^2 \; \left(\partial_t \rho + \vec{u} \cdot \vec{\nabla} \rho_0\right),$$

Adiabatic sound waves in a homogeneous sphere II

- Time-harmonic solutions $P(\vec{x},t) = \Re[\hat{P}(\vec{x})\exp(-i\omega t)]$ of the wave equation $\Delta P - \frac{1}{c_s^2}\frac{\partial^2 P}{\partial t^2} = 0$ read :

$$\Delta \hat{P} + \frac{\omega^2}{c_s^2}\hat{P} = 0$$

- In a spherical coordinate system $[r, \theta, \phi]$, separable solutions $\hat{P}(\vec{x}) = \frac{u(r)}{r}\Theta(\theta)\Phi(\phi)$ exist and verify :

$$\Phi(\phi) \propto \cos(m\phi + \phi_0), \quad \Theta(\theta) \propto \mathrm{P}_\ell^m(\cos\theta), \quad \frac{d^2 u}{dr^2} + k^2 u = 0$$

where $\mathrm{P}_\ell^m(z)$ is a Legendre polynomia, m and ℓ are integers and $k^2(r) = \frac{\omega^2}{c_s^2} - \frac{\ell(\ell+1)}{r^2}$.

- The pressure perturbation vanishes at the surface $\Rightarrow$

$$\omega = j_{n,\ell+1/2}\frac{c_s}{R}$$

where $j_{n,\ell+1/2}$ is the n^{th} root of the Bessel function $J_{\ell+1/2}$.

Adiabatic sound waves in a homogeneous sphere III

- A discrete spectrum : $\omega_{n,\ell,m} = j_{n,\ell+1/2}\frac{c_s}{R}$
- In the limit $\frac{n}{\ell} \to \infty$, $\omega_{n,\ell,m} = \pi\frac{c_s}{R}(n+\ell/2) + O(1/\omega)$
- Asymptotic frequency spacings :
  $\Delta_n = \omega_{n+1} - \omega_n = \pi\frac{c_s}{R}$ and $\Delta_\ell = \omega_{\ell+1} - \omega_\ell = \frac{\Delta_n}{2}$

Governing equation

- 3D Helmholtz equation :
  $\Delta u + k^2 u = 0$, k constant
- 1D reduced Helmholtz type equation :
  $\frac{d^2u}{dr^2} + k(r)^2 u = 0$

Spectrum properties

- Regular spectrum
  $\omega = f(n, \ell, m)$
- Asymptotically $\Delta_n, \Delta_\ell, \Delta_m$ are constant.

Outline

Adiabatic oscillations in a non-rotating star under the Cowling approximation

Small perturbations about the equilibrium stellar model verify :

$$\partial_t \rho + \vec{\nabla} \cdot (\rho_0 \vec{u}) = 0,$$

$$\rho_0 \partial_t \vec{u} = -\vec{\nabla} P + \rho \vec{g}_0,$$

$$\partial_t P + \vec{u} \cdot \vec{\nabla} P_0 = c_s^2 \left(\partial_t \rho + \vec{u} \cdot \vec{\nabla} \rho_0 \right),$$

where $\vec{u}$, ρ and P, are respectively the Eulerian perturbations of velocity, density and pressure, $c_s = \sqrt{\Gamma P_0/\rho_0}$, and $\vec{g}_0$ is the gravity.

Reduction to a 1D Helmholtz type equation

Looking for time-harmonic separable solutions, the full 3D boundary value problem is reduced, for each value of ℓ and m, to a 1D Helmholtz type boundary value (Gough, 1993):

$$\frac{d^2u}{dr^2} + k^2(r)u = 0$$

where u is such that $\delta p = \left(\frac{g_0 \rho_0 f_0}{r^3}\right)^{1/2} u(r)\Re[Y_\ell^m(\theta,\phi)\exp(-i\omega t)]$

$$k^2(r) = \frac{\omega^2 - \omega_c^2}{c_s^2} - \frac{\ell(\ell+1)}{r^2}\left(1 - \frac{\mathcal{N}}{\omega^2}\right)$$

where f_0, $\mathcal{N}$ and ω_c are defined by:

$$f_0 = \frac{\omega^2 r}{g_0} + 2 + \frac{r}{H_g} - \frac{\ell(\ell+1)g_0}{r\omega^2}, \quad \mathcal{N}^2 = g_0\left(\frac{1}{\mathcal{H}} - \frac{g_0}{c_s^2} - \frac{2}{h}\right)$$

$$\omega_c^2 = \frac{c_s^2}{4\mathcal{H}}\left(1 - 2\frac{d\mathcal{H}}{dr}\right) - \frac{g_0}{h} \quad \text{where } h^{-1} = H_g^{-1} + 2r^{-1}$$

$H, H_g, H_f, \mathcal{H}$ being the scale height of $\rho_0, g_0, f_0, \left(\frac{g_0\rho_0 f_0}{r^3}\right)^{1/2}$, respectively.

Other systems reduced to $\frac{d^2u}{dr^2} + k^2(r)u = 0 \quad +$ B.C.

- Quantum physics : Stationary state of particle in a spherically symmetric potential where
  - $\Psi(\vec{x}, t) = \frac{u(r)}{r} Y_{\ell m}(\theta, \phi) \exp(-i\frac{E}{\hbar}t)$ is the wavefunction
  - E is the energy level
  - $k^2(r) = \frac{2m}{\hbar^2}[E - V(r)] - \frac{\ell(\ell+1)}{r^2}$ depends on the potential $V(r)$
- Optics : Electromagnetic waves in a infinite cylinder (linear, isotropic, and non-dispersive material)
  - $E_r(\vec{x}, t) = \frac{u(r)}{r^{1/2}} \exp\left[i(m\phi + k_z z - \omega t)\right]$ is the radial electric field
  - $k^2(r) = \frac{n^2\omega^2}{c^2} - \frac{m^2 - 1/4}{r^2} - k_z^2$
  - ω is the frequency, c the speed of light, n the medium index

Except in very simple systems like the homogeneous sphere, there is no analytical solutions to this equation

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The WKB approximation to $\frac{d^2y}{dx^2} + k^2(x)y = 0$

- If k is a constant $\Rightarrow$ $y \propto \exp(ikx)$
- If k is not constant, in a region where k does not vary rapidly, let's try a wave-like solution :

$$\Psi(x) \propto \exp\left[i \int k(x')dx'\right]$$

$$\frac{d\Psi(x)}{dx} = ik(x)\Psi(x) \quad \Rightarrow \quad \frac{d^2\Psi(x)}{dx^2} = i\frac{dk(x)}{dx}\Psi(x) - k(x)^2\Psi(x)$$

- This will constitute an approximate solution if

$$\frac{dk(x)}{dx} \ll k(x)^2, \quad \text{or} \quad \frac{1}{k} \ll \frac{k}{\frac{dk(x)}{dx}}$$

that is if the wavelength of the wave-like solution $\lambda = \frac{2\pi}{k}$ is much smaller than the typical lengthscale of variation of the background medium, $H_k = \frac{k}{\frac{dk(x)}{dx}}$.

The WKB approximation II

- The $i\frac{dk(x)}{dx}\Psi(x)$ term can be compensated by introducing a slowly varying wave amplitude $A(x)$, that is such that:

$$\Psi(x) \propto A(x)\exp\left[i\int k(x')dx'\right] \quad \text{where} \quad \frac{1}{k} \ll \frac{A}{\frac{dA(x)}{dx}}$$

- Below the $\mathcal{O}\left(\frac{H_k^2}{\lambda^2}\right)$ terms, we have :

$$\frac{d^2A}{dx^2} + 2i\frac{dA}{dx}k + iA\frac{dk}{dx} = 0$$

- The $\mathcal{O}\left(\frac{H_k}{\lambda}\right)$ terms yield :

$$2\frac{dA}{dx}k + A\frac{dk}{dx} = 0 \quad \text{that is} \quad A(x) \propto \frac{1}{\sqrt{k(x)}}$$

The WKB approximation III

- The solution is thus:

$$y(x) = \begin{cases} A_0 \frac{1}{k(x)^{1/2}} \cos\left(\int_{x_0}^{x} k(x')dx' + \phi\right) & \text{if } k^2 > 0 \\ A_0 \frac{1}{|k(x)|^{1/2}} \exp\left(\pm \int_{x_0}^{x} |k(x')|dx'\right) & \text{if } k^2 < 0 \end{cases} \quad (1)$$

For such wave-like solution the (variable) wavevector is $\vec{k}(x) = k(x)\vec{e}_x$ is the derivative of the phase $\Phi(x) = \int_{x_0}^{x} k(x')dx'$.

According to $\frac{1}{k} \ll \frac{k}{\frac{dk(x)}{dx}}$, the WKB approximation breaks down when

- the background medium varies on a lengthscale comparable to or smaller than the wavelength of the solution
- $k(x) \rightarrow 0$ as the wavelength becomes infinite

The turning points $k^2(r) = 0$

- For p-modes, r_i and r_e are such that $\frac{\omega}{c_s(r_i)} \approx \frac{\sqrt{\ell(\ell+1)}}{r_i}$ and $\omega \approx \omega_c(r_e)$.
- Near the turning point x_i, $k^2(x) \approx \alpha(x - x_i)$ and the reduced Helmholtz eq. is close to $\frac{d^2y}{dz^2} = -zy$, where $z = \alpha^{1/3}(x - x_i)$.
- The general solution is $y(z) = C_1\mathrm{Ai}(z) + C_2\mathrm{Bi}(z)$ where Ai and Bi are the Airy functions. For large negative z, $\mathrm{Ai}(z) \approx \frac{1}{2\pi^{1/2}}|z|^{-1/4} \exp\left(-\frac{2}{3}|z|^{3/2}\right)$ while $\mathrm{Bi} \to \infty$. For large positive z, $\mathrm{Ai}(z) \approx \frac{1}{\pi^{1/2}}|z|^{-1/4} \cos\left(\frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right)$.

$$\Rightarrow \quad C_2 = 0 \;\text{ and }\; \int_{x_i}^{x} k(x')dx' + \phi = \frac{2}{3}|z|^{3/2} + \phi \quad \Rightarrow \quad \phi = -\frac{\pi}{4}$$

The asymptotic formula I

- Away from x_i, $y(x) = A_1 |k(x)|^{-1/4} \cos\left(\int_{x_i}^{x} k(x')dx' - \frac{\pi}{4}\right)$
- Away from x_e, $y(x) = A_2 |k(x)|^{-1/4} \cos\left(\int_{x}^{x_e} k(x')dx' - \frac{\pi}{4}\right)$
- The condition that the two solutions be equal and continuous at an intermediary point such that: $\Psi_1 = \left(\int_{x_i}^{x_f} k(x')dx' - \frac{\pi}{4}\right)$ and $\Psi_2 = \left(\int_{x_f}^{x_e} k(x')dx' - \frac{\pi}{4}\right)$ yields:

$$A_1 \cos\Psi_1 = A_2 \cos\Psi_2 \quad \text{and} \quad -A_1 \sin\Psi_1 = A_2 \sin\Psi_2$$

- The system has non-trivial solution for A_1 and A_2 if:

$$\sin\Psi_1 \cos\Psi_2 + \sin\Psi_2 \cos\Psi_1 = \sin(\Psi_1 + \Psi_2) = 0 \quad \Rightarrow \quad \Psi_1 + \Psi_2 = (n-1)\pi$$

$$\int_{x_i}^{x_e} k(x')dx' = \left(n - \frac{1}{2}\right)\pi$$

The asymptotic formula II

$$\int_{r_i}^{r_e} \left[\frac{\omega^2 - \omega_c^2}{c_s^2} - \frac{L^2}{r^2} \left(1 - \frac{\mathcal{N}}{\omega^2} \right) \right]^{1/2} dr = \left(n - \frac{1}{2} \right) \pi + \Psi_0$$

where $L^2 = \ell(\ell+1)$ and Ψ_0 accounts for the B.C. at the star surface.

- For p-modes, the term $\mathcal{N}/\omega$ is small :

$$\frac{\pi(n+\alpha)}{\omega} = \int_{r_i}^{r_e} \left(1 - \frac{\omega_c^2}{\omega^2} - \frac{L^2 c_s^2}{\omega^2 r^2} \right)^{1/2} \frac{dr}{c_s} \quad \text{where } \alpha = \Psi_0/\pi - 1/2$$

- In the limit $n/\ell \to \infty$, the Tassoul's formula is recovered:

$$\omega = \frac{\pi}{\int_0^{r_e} \frac{dr}{c_s}} \left(n + \frac{1}{2} L + \alpha \right) + \mathcal{O}(1/\omega)$$

Governing equation

- Adiabatic oscillation, Cowling assumption
- 1D reduced Helmholtz type equation : $\frac{d^2 u}{dr^2} + k(r)^2 u = 0$

Spectrum properties

- Regular spectrum $\omega = f(n, \ell, m)$
- Asymptotically $\Delta_n, \Delta_\ell, \Delta_m$ are constant.

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Wave equation/ray dynamics

- Electromagnetic waves/ Geometrical optics
- Acoustic waves/ Acoustic ray
- Quantum / Classical mechanics

In all these cases, the description of the wave propagation by a ray path (or a trajectory of a dynamical system) is obtained through a WKB approximation of the 3D wave equation. It is an asymptotic description as the wavelength is assumed to be much smaller that the lengthscale of variation of the background medium.

The WKB approximation of the 3D oscillation equations in a rotating star

The equations governing adiabiatic oscillations where the perturbation of the gravitional potential and the Coriolis force are neglected ($\vec{g}_0 = -\vec{\nabla}\left(\psi_0 - \Omega^2 w^2/2\right)$ is the effective gravity which includes the centrifugal force) read:

$$\partial_t \rho + \vec{\nabla} \cdot (\rho_0 \vec{u}) = 0,$$

$$\rho_0 \partial_t \vec{u} = -\vec{\nabla} P + \rho \vec{g}_0,$$

$$\partial_t P + \vec{u} \cdot \vec{\nabla} P_0 = c_s^2 \left(\partial_t \rho + \vec{u} \cdot \vec{\nabla} \rho_0\right),$$

They can be reduced to :

$$\frac{\omega_c^2 - \omega^2}{c_s^2} \hat{\Psi} + \frac{N_0^2}{\omega^2} [\Delta - \frac{1}{g_0^2} (\vec{g_0} \cdot \vec{\nabla})(\vec{g_0} \cdot \vec{\nabla})] \hat{\Psi} = \Delta \hat{\Psi}$$

where $\hat{\Psi} = \hat{P}/\alpha$, and α a function of the equilibrium model quantities.

The WKB approximation of the 3D oscillation equations in a rotating star

- We seek wake-like solutions under the WKB approximation:

$$\Psi = \Re\{A(\vec{x}) \exp[i\Phi(\vec{x}) - i\omega t]\}$$

- If Λ denotes the ratio between the model scale height and the solution wavelength:

$$\Phi = \Lambda(\Phi_0 + \frac{1}{\Lambda}\Phi_1..) \quad A = A_0 + \frac{1}{\Lambda}A_1..$$

- The highest $\mathcal{O}(\Lambda^2)$ terms yields:

$$\frac{\omega^2 - \omega_c^2}{c_s^2} + \frac{N_0^2}{\omega^2}\Lambda^2(\vec{\nabla}\Phi_0)_\perp^2 = \Lambda^2(\vec{\nabla}\Phi_0)^2$$

where $(\vec{\nabla}\Phi_0)_\perp = \vec{\nabla}\Phi_0 - (\vec{\nabla}\Phi_0 \cdot \vec{n}_0)\vec{n}_0$ and $\vec{n}_0$ is the outward unit vector in the direction opposite to the effective gravity.

- For high frequency acoustic waves $\frac{\omega}{\omega_0} = \mathcal{O}(\Lambda)$ which implies:

$$\frac{\omega^2 - \omega_c^2}{c_s^2} = (\vec{\nabla}\Phi_0)^2$$

This is the eikonal equation.

The acoustic ray dynamics

- We look for solutions of the eikonal equation (a PDE):

$$\frac{\omega^2 - \omega_c^2}{c_s^2} = (\vec{\nabla}\Phi)^2$$

along some path called the ray path. We consider a path normal to the wavefront $\Phi(\vec{x}) = const.$, that is following the wavevector $\vec{k} = \vec{\nabla}\Phi$:

$$\frac{d\vec{x}}{ds} = \frac{\vec{k}}{\|\vec{k}\|} = \frac{\vec{\nabla}\Phi}{\|\vec{\nabla}\Phi\|}$$

where s is the curvilign abscissa along the ray.

- This leads to :

$$\frac{d\vec{x}}{dt} = \vec{\tilde{k}}, \quad \frac{d\vec{\tilde{k}}}{dt} = -\vec{\nabla}W, \quad \text{where } W = -\frac{1}{2c_s^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)$$

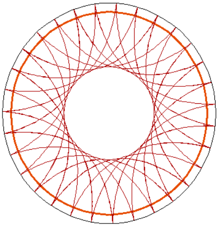
where we use the frequency-scaled wavevector $\vec{\tilde{k}} = \vec{k}/\omega$ and the time-like variable t such that $dt = c_s ds/(1 - \omega_c^2/\omega^2)^{1/2}$. Both equations constitute Hamilton equations for the Hamiltonian :

$$H = \frac{\tilde{k}^2}{2} + W(\vec{x})$$

Ray dynamics in a spherical star

Due to the symmetries of the system, the Hamiltonian is integrable:

- The phase space dimension is equal to $6 = 3$ position coordinates $+3$ momentum coordinates
- The number of degree of freedom is thus $6/2 = 3$
- There are three invariants: $L_z = r \sin\theta k_\phi$ (axial symmetry), $L = r(k_\theta^2 + k_\phi^2)^{1/2}$ (spherical symmetry) and the Hamiltonian H.



All trajectories (except $L = 0$ trajectories) are of the whispering gallery type

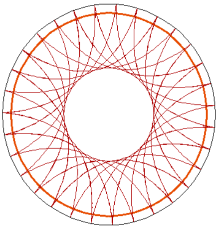
How can we construct modes from the acoustic rays ?

- The phase difference between two points on a ray $\mathcal{T}$ reads:

$$\Delta\Phi = \int_{\mathcal{T}} d\Phi = \int_{\mathcal{T}} \vec{\nabla}\Phi \cdot d\vec{x} = \int_{\mathcal{T}} \vec{k} \cdot d\vec{x}$$

- When the propagating wave intersects itself, the phase accumulated along the ray must be a multiple of 2π otherwise the WKB solution $A(\vec{x})\exp[i\Phi(\vec{x})]$ is not single-valued :

$$\int_a^b \vec{k} \cdot d\vec{x} = 2n\pi$$



Superposed travelling waves must interfere constructively.

The EBK semiclassical quantization

How this quantization condition can be applied in pratice depends on the nature of the Hamiltonian system :

- If the system is integrable, the trajectories stay on a torus like surface in phase space $\Rightarrow$ the Einstein (1917), Brillouin (1926) and Keller (1958) condition:

  $$\int_{C_j} \vec{k} \cdot \vec{dx} = 2\pi(n_j + \frac{\beta_j}{4})$$

  where C_j is any closed contour on the torus and the integer β_j called the Maslov index is introduced to account for a $\pi/2$ phase lag that must added each time the contour crosses a caustic.

- If not, it's more complicated ... (see the Guztwiller trace formula)

Simple applications of the EBK semiclassical quantization I

- The vibrating string :

$$\int_a^b k ds = 2n\pi \quad \Rightarrow \quad k \times 2L = 2n\pi$$

- The homogeneous sphere $\omega = c_s\sqrt{k_r^2 + k_\theta^2 + k_\phi^2}$
  Three closed contours on the L_z, L, ω torus
  - $\int_{C_\phi} \vec{k} \cdot \vec{dx} = \int_0^{2\pi} k_\phi r \sin\theta d\phi = \int_0^{2\pi} L_z d\phi = 2\pi L_z = 2\pi m$
  - $\int_{C_\theta} \vec{k} \cdot \vec{dx} = \int_{C_\theta} k_\theta r d\theta = \int_{C_\theta} \left(L^2 - \frac{L_z^2}{\sin^2\theta}\right)^{1/2} d\theta =$
    $2\pi(L - L_z) = 2\pi(\ell' + 1/2)$
  - $\int_{C_r} \vec{k} \cdot \vec{dx} = \int_{C_r} k_r dr = 2\pi(n' + 1/2)$

$$\int_{r_1}^{r_2} \left(\frac{\omega^2}{c_s^2} - \frac{(\ell + 1/2)^2}{r^2}\right)^{1/2} dr = \pi(n - 1/2)$$

where $\ell = \ell' + m$, and $n = n' + 1$.

EBK semiclassical quantization in a spherical star

Invariants L_z, L, ω and contours are the same as for the homogeneous sphere :

- $L_z = m$
- $L = \ell + 1/2$
- $$\int_{r_i}^{r_e} \left[\frac{\omega^2 - \omega_c^2}{c_s^2} - \frac{L^2}{r^2} \left(1 - \frac{\mathcal{N}}{\omega^2} \right) \right]^{1/2} dr = \left(n - \frac{1}{2} \right) \pi$$

Conclusion

The result is very similar to the usual asymptotic theory but the ray dynamics formalism applies to a wider range of problems.