Asymptotic structure of the acoustic frequency spectrum II

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Objectives of this lecture

- Present the basic tools necessary to derive the asymptotic formulas in spherical stars
- Present an alternative method based on acoustic ray dynamics
- Show that non-regular spectra exist in more complex systems
- Present current research on the asymptotic theory in rotating stars

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Outline I

- What motivates an asymptotic analysis of stellar pulsation modes ?
- Regular frequency spectra
- Irregular spectra
  - The frequency statistics of classically chaotic system
  - Experimental signatures of quantum chaos
- The case of uniformly rotating polytropic stars
  - Integrable or chaotic ray dynamics ?
  - Interpretation of the ray dynamics in terms of mode properties

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Regular versus irregular spectra

- For wave system whose small wavelength limit is integrable, the EBK quantization enables to find a function f such that :

  $\omega = f(n_1, n_2, ..., n_N)$ where N is degree-of-freedom of the Hamiltonian

  The spectrum is said to be regular

- This is not the case for wave system whose small wavelength limit is chaotic and the resulting spectrum is said to be irregular

Quantum chaos looks for the features of a quantum system that are related to the chaotic nature of its classical limit.

The frequency (or energy level) statistics

The statistics of consecutive energy level spacings $\Delta_i = E_{i+1} - E_i$

- The averaged level separation, $\Delta E = \sum_{i=1}^{N} \frac{E_{i+1}-E_i}{N}$, depends on the global geometrical properties of the system as well as on the energy range considered.
- Δ_i fluctuates about this average and the statistics of these fluctuations appears to depend on the nature, integrable or chaotic, of the underlying classical dynamics
- Histograms of the scaled level spacing, $\delta_i = \frac{E_{i+1}-E_i}{\Delta E}$, enables to study these fluctuations

- Classicaly integrable system : $P(\delta) = \exp(-\delta)$, Poisson distribution
- Classicaly chaotic system : $P(\delta) \approx \frac{\pi}{2}\delta \exp(-\pi\delta^2/4)$, Wigner distribution
    - The Wigner distribution follows from Wigner's hypothesis that the statistical properties of the eigenvalues of ensembles of random matrices should be similar to that of the spectra of complicated nuclear systems
    - The matrix elements follow a Gaussian distribution

The distributions are

- generic
- parameter-free

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1. Irregular spectra
   - The frequency statistics of classically chaotic system
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Quantum billiard

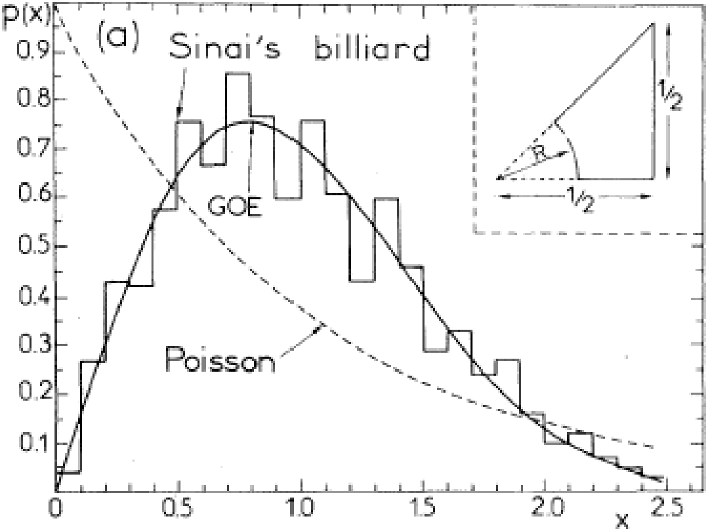
Stationary state of a particle in a potential well: $V = 0$ inside, $V = \infty$ on the boundary

- $\Psi(\vec{x}, t) = U(\vec{x}) \exp(-i \frac{E}{\hbar} t)$ is the wavefunction
- E is the energy level
- $\Delta U + k^2 U = 0$ where $k^2 = \frac{2m}{\hbar^2} E \quad + B.C.$ on the billiard boundary

The classical dynamics of most billiards is not integrable, some billiards being fully chaotic.

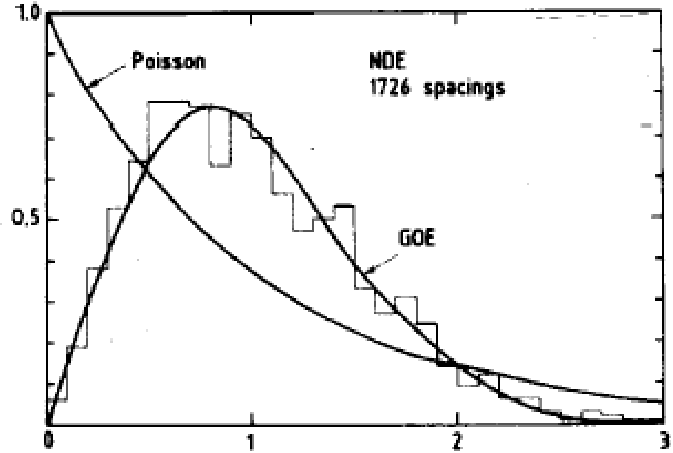
Chaotic Quantum billiard

Bohigas et al., 1984



Quantum physics experiment - Nuclear energy levels

Bohigas et al., 1983



From 1726 spacings corresponding to 30 energy sequences of 27 different nuclei

Quantum physics experiment - Atomic level statistics
Camarda & Georgopulos 1983

VOLUME 50, NUMBER 7 PHYSICAL REVIEW LETTERS 14 FEBRUARY 1983

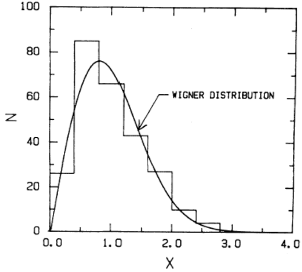


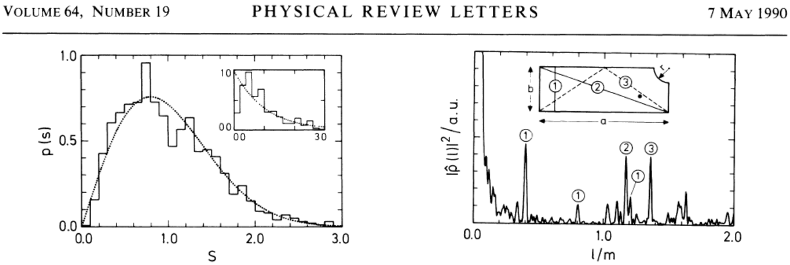
FIG. 3. Combined adjacent-level spacing distribution of eight sets of atomic energy levels. The smooth curve represents the Wigner distribution which accurately reflects a prediction of random-matrix theory.

adjacent levels are correlated), the average value of $\mathrm{Cov}(S_i, S_{i+1})$ for our eight samples is -0.23 with a standard deviation 0.054. This is within one standard deviation of the GOE prediction of -0.27 and more than four standard deviations from zero. The agreement with the GOE is good. Listed in the last three columns of Table I are values of the Δ_3 statistic found experimentally, predicted by GOE, and expected for UW levels. Agreement between the expected GOE Δ_3 values and those found experimentally is quite good. The expected Δ_3 values of the UW levels differ "greatly" from the experimental values, but their individual uncertainty is large. It is significant, however, that all experimental Δ_3 values lie below the expected UW values. We construct the statistic $\Delta = \sum_i \Delta_{3i}$ and find that $\Delta^{\mathrm{exp}} = 2.98$, $\Delta^{\mathrm{GOE}} = 2.77 \pm 0.31$, and $\Delta^{\mathrm{UW}} = 5.51 \pm 1.12$. Δ^{exp} is less than one standard deviation

From about 270 levels of eight different rare-earth atoms

Wave chaos - Microwaves resonators

Stöckmann & Stein 1990



From 1002 eigenfrequencies in the frequency range 1-18 GHz

Wave chaos - Acoustic resonances in quartz blocks
Ellegaard et al. 1996

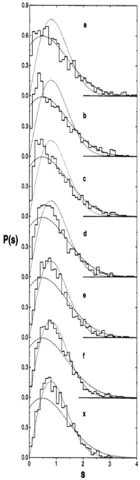


FIG. 2. The nearest neighbor spacing distributions $P(s)$ for the different radii r of the removed octant: (a) $r = 0$, the flip symmetry is fully conserved, (b) $r = 0.5$ mm, (c) $r = 0.8$ mm, (d) $r = 1.1$ mm, (e) $r = 1.4$ mm, (f) $r = 1.7$ mm, (x) the block with the huge defocusing structures, these data were derived from a spectrum ranging from 720 to 920 kHz. The dotted and the dashed curves are the theoretical predictions for a chaotic system containing no or one symmetry, respectively.

Symmetry breaking effect - From about 1400 resonnances

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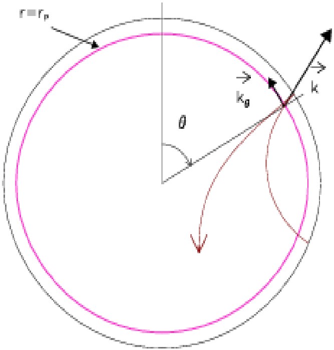
Integrable or chaotic dynamics ?

Computation of the ray dynamics

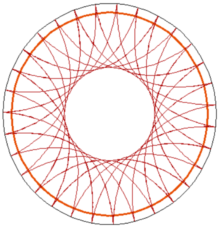
$$\frac{d\vec{x}}{dt} = \vec{\tilde{k}}, \quad \frac{d\vec{\tilde{k}}}{dt} = -\vec{\nabla} W, \quad \text{w here } W = -\frac{1}{2c_s^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)$$

- Using the axial symmetry about the rotation axis, the 3 degree-of-freedom Hamiltonian is reduced to a 2 degree-of-freedom Hamiltonian which depends on L_z as a parameter
- The uniformly rotating polytropic model of star is computed numerically for increased values of Ω/Ω_K where $\Omega_K = \left(\frac{GM}{R^3}\right)^{1/2}$ is the breakup rotation rate
- For each stellar model, ray trajectories are computed for many different initial conditions using Runge-Kutta method

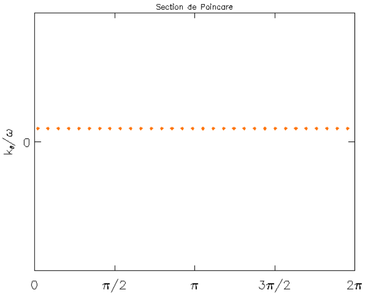


- θ and k_θ/ω specify a point on the Poincaré Surface of Section
- the 4-dimensional phase space $[r, \theta, k_\theta, k_r]$ is vizualized through the 2-dimensional Poincaré Surface of Section

Ray dynamics - Phase space visualization
The spherical case



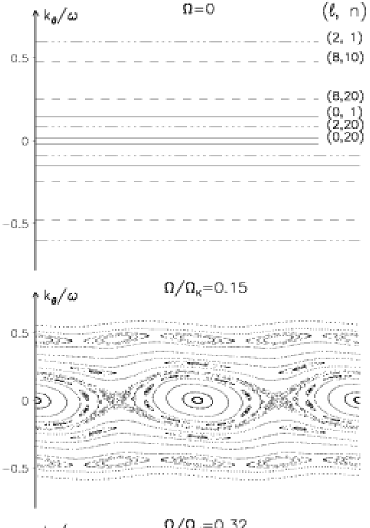
Trajectory

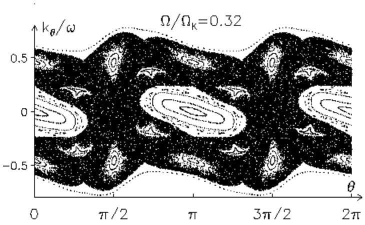


Poincaré Surface of Section

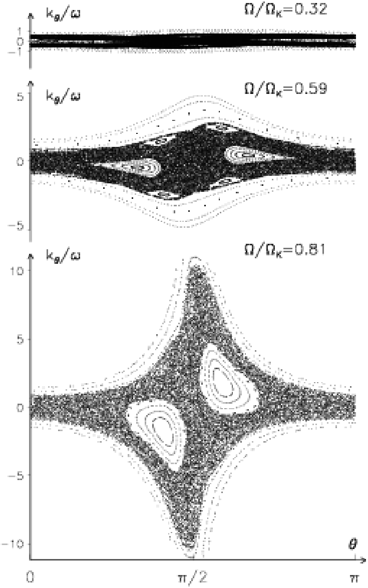
Ray dynamics - The transition to chaos

"Small" rotation rates





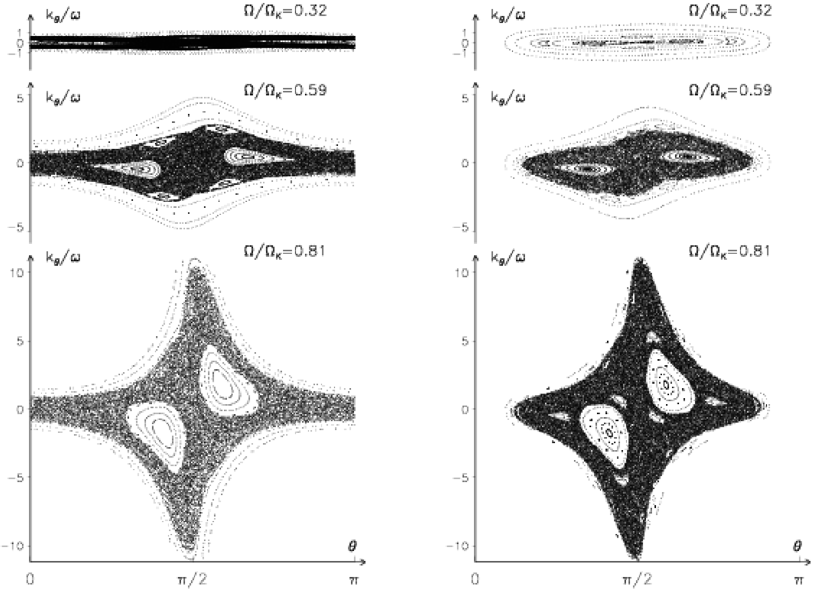
- smooth transition following the Kolmogorov-Arnold-Moser theorem
- the structures of the integrable phase space (the invariant tori) are progressively destroyed
- chaotic regions and new complex structures develop



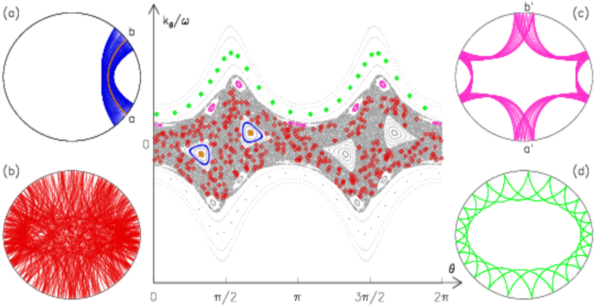
- the central chaotic region enlarges
- the 2-period island chain persists
- smaller island chain get progressively destroyed

Ray dynamics - The transition to chaos

"High" rotation rates



Trajectories and phase space
$\Omega/\Omega_K = 0.6$



Conclusion on the acoustic ray dynamics in rapidly rotating stars

- Island chains, surviving tori and chaotic regions coexist
- The Hamiltonian system is said to be mixed

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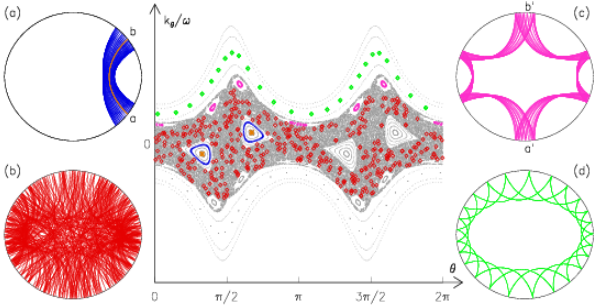

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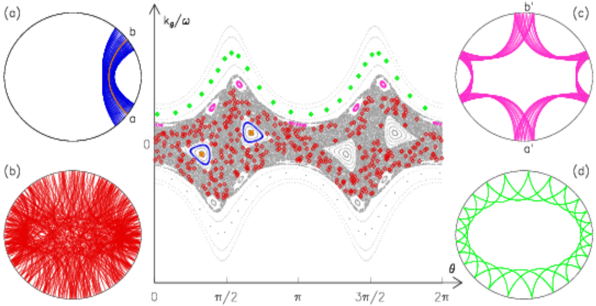

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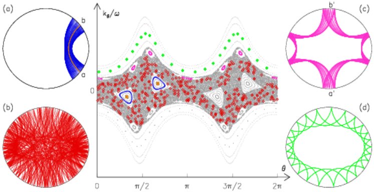
Interpretation of the ray dynamics in terms of mode properties



The Percival conjecture for mixed systems

- Modes constructed in a dynamically-independent phase space region form an independent family with specific properties
- Modes constructed from a near-integrable region (island chains, surviving tori) yield a regular spectrum
- Modes constructed from a chaotic region yield an irregular spectrum
- The resulting spectrum is a superposition of regular and irregular sub-spectra

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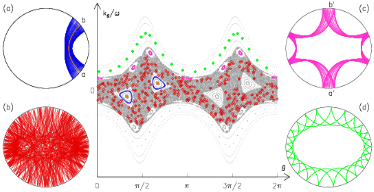

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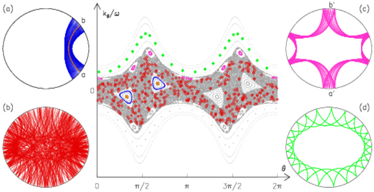

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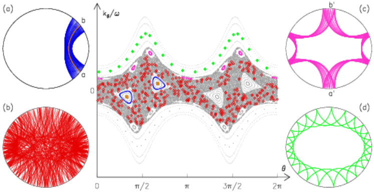

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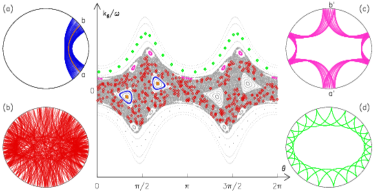

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- Computation of high frequency p-modes : $20 < n < 30$
- The mode is not separable in the latitudinal and radial direction
  $\Rightarrow$ A 2D eigenvalue problem has to be solved
- After discretization $\Rightarrow$ Eigenvalues of large (very large) matrices
- We used spectral methods to minimize the matrix size and a surface-fitting coordinate system for accurate boundary conditions. For the present p-modes, the matrix has N^2 elements where $N = N_r \times N_\theta$ and $N_r = 100$, $N_\theta = 150$ (spherical harmonics from $\ell = 0$ up to $\ell = 150$)

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Confrontation with numerically computed modes I

Mode computation

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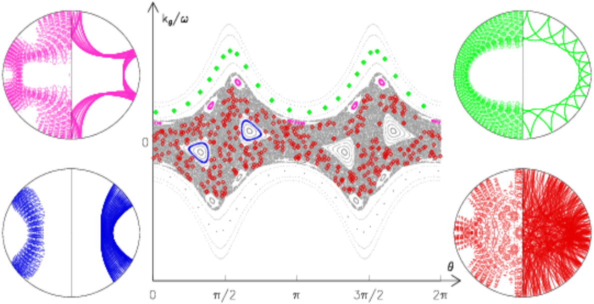
Confrontation with numerically computed modes II

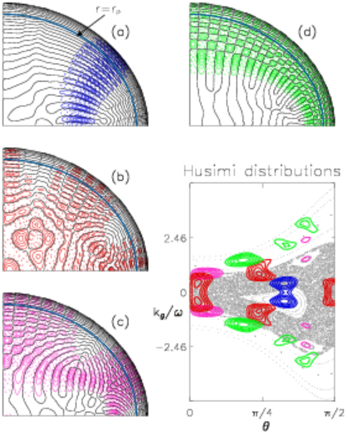
Confrontation in physical space



Comparison between modes and acoustic rays in physical space

Confrontation with numerically computed modes III

Confrontation in phase space $\Rightarrow$ mode classification



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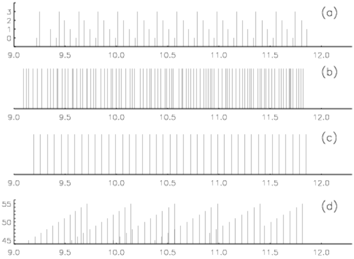


The four sub-spectra resulting from the phase space classification

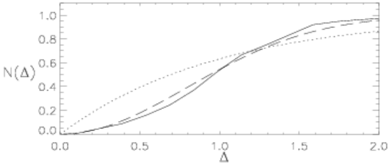
Properties of the sub-spectra

- The 2-period island modes and the 6-period island modes follow simple empirical laws:

$$\omega_{n\ell} = n\delta_n + \ell\delta_\ell + \alpha$$

$$\omega_{n\ell} = n\delta'_n$$

- The whispering gallery mode do not follow this simple law. They can be simply characterized by the number of nodes in the latitudinal and pseudo-radial directions
- The frequency statistics of the ≈ 200 chaotic modes is compatible with the Wigner distribution



The system follows the Percival conjecture

Quantitative results of the asymptotic theory

- The EBK quantization of near-integrable regions
  - The 2-period island modes : $\omega_{n\ell} = n\delta_n + \ell\delta_\ell + \alpha$

$$\delta_n = \pi/(\int_{\mathrm{a}}^{\mathrm{b}} d\sigma/c_s) \quad \delta_\ell = 2(\int_{\mathrm{a}}^{\mathrm{b}} c_s d\sigma/w^2)/(\int_{\mathrm{a}}^{\mathrm{b}} d\sigma/c_s)$$

$$\delta_n = 0.5635\sqrt{GM/R_p^3} \text{ within 2\% of the empirical value}$$

  - The 6-period island modes : $\omega_{n\ell} = n\delta'_n$

$$\delta'_n = \pi/(\int_{\mathrm{a'}}^{\mathrm{b'}} d\sigma/c_s)$$

$$\delta'_n = 0.186\sqrt{GM/R_p^3} \text{ within 3\% of the empirical value}$$

- The frequency statistics of the chaotic mode spectrum
- According to the Weyl formula, the averaged number of modes in a phase space region is given by its phase space volume

Synthetic spectra can be constructed from the asymptotic theory

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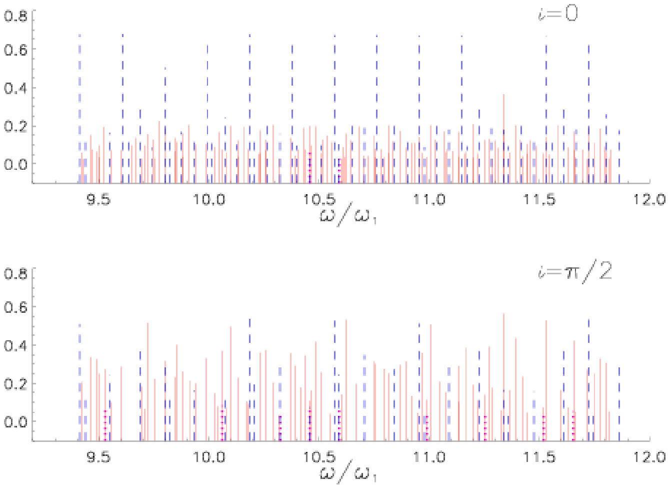
Towards identification tools



The regularities of the 2-period island modes are hidden by the chaotic mode frequencies

Conclusion

- Limitations of the asymptotic theory
- More realistic model of rapidly rotating stars
- Observational evidences of the Wigner distribution ? Pole-on stars ?
- Extension to gravity waves in rapidly rotating stars ?

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