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Spin-Orbit Coupling in the Solar System

PETER GOLDREICH AND STANTON PEALE
Department of Astronomy and Institute of Geophysics and Planetary Physics, University of California at Los Angeles, Los Angeles, California

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Two possible types of resonant spin rates for planets and satellites are investigated. The first occurs in eccentric orbits at rotation rates which are commensurate with the orbital mean motion. A resonant spin state exists at each half-integer multiple of the mean motion, the simplest case being the well-known synchronous rotation. The second class of resonant spins involves the presence of another planet or satellite. A planet (or satellite) with such a resonant spin always aligns the same axis toward the second planet (or satellite) at each conjunction.

Averaged equations of motion are derived, and stability criteria are formulated for both types of resonance. Probabilities of capturing a planet (or satellite) into one of the commensurate rotation states as it is being despun by tidal friction are calculated.

Application of the results to Mercury reveals that the very small value of $(B\text{-}A)/C \simeq 10^{-8}$ would suffice to stabilize Mercury's rotation period at $\frac{2}{3}$ of its orbital period. The probability that Mercury would be captured at this resonance is calculated for several assumed forms of tidal torques. Venus may be in a resonant spin state of the second kind. A sidereal rotation period of 243.16 days retrograde would be commensurate with its synodic motion. However, a large value of $(B-A)/C (\geq 10^{-4})$ seems to be required to stabilize this rotation. In addition, the capture probability at this resonance appears to be small.

I. INTRODUCTION

SEVERAL theoretical discussions of planetary and satellite rotation rates have followed the recent radar determination of Mercury's rotation period (Pettengill and Dyce 1965). (Hereafter we only refer to planets, although our results may be applied to satellites as well.) In the first, Peale and Gold (1965) showed that in an eccentric orbit tidal friction could bring an axially symmetric planet to an asymptotic rotation rate which is somewhat faster than its orbital mean motion. In the asymptotic spin state the tidal torque averaged over an orbit period vanishes. For a broad class of tidal torques the maximum torque occurs at perihelion, and the final spin velocity will lie between the mean motion and the instantaneous orbital angular velocity at perihelion. The precise value of the final spin is determined by the amplitude and frequency dependence of the planet's Q, where $1/Q$ is the specific dissipation function (MacDonald 1964). In Fig. 1 we have displayed the asymptotic rotation rate of an axially symmetric Mercury as a function of eccentricity for four types of tidal torques, where δ is the angle by which the high tide leads the sun. We discuss the different forms of tidal torque in Sec. III.

Colombo (1965) has suggested that Mercury may be spinning with an angular velocity of precisely $1.5n$, where n is its orbital mean motion. Subsequent investigations have shown that a sufficient deviation from axial symmetry would stabilize this resonant spin (Liu and O'Keefe 1965; Colombo and Shapiro 1965; Goldreich and Peale 1966a). The existence of other resonant spin states at rotation rates of pn, where p is any half-integer (negative or positive), has been suggested by the latter two sets of authors. We have shown that the exceedingly small value of $(B-A)/C \approx 10^{-8}$ would suffice to stabilize many of these resonances against the disruptive influence of the solar tidal torque. Here, A, B, and C are the principal moments of inertia in order of increasing magnitude. For the moon in its present orbit with the known value of $(B-A)/C \simeq 2\times10^{-4}$, stable resonant spin states exist at rotation rates of $0.5n$, n, $1.5n$, $2n$, $2.5n$ and perhaps at several others. Thus for Mercury and the moon, stable spin states appear to exist with rotation rates which are both faster and slower than the observed values. Almost certainly Mercury and the moon must have bypassed some of these stable resonances on the way to attaining their present spins. Similar strong indications that satellites may pass through stable resonances are provided by the observed synchronous rotations of several satellites of the major planets. Of special interest is Iapetus, because of its weak tidal torque, relatively high orbital eccentricity, and well-confirmed synchronous rotation (Widorn 1950).

What conditions must be satisfied if a planet is to be captured at one of the resonant states? The present investigation is devoted to answering this question.

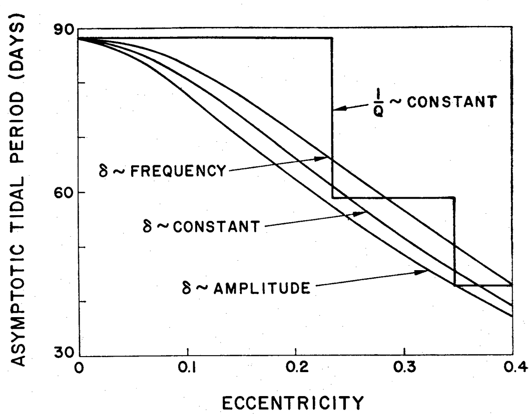

FIG. 1. Mercury's asymptotic tidal spin period for four types of tidal torque.

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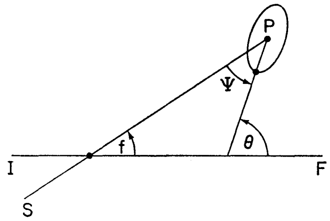

FIG. 2. Coordinates and angles used in discussion of first type of resonance.

In the following, approximate equations of motion for a spinning planet are derived by averaging the complete equation of motion over an orbit period. From the averaged equations criteria for the stability of resonant spin states are established. Details of the capture into stable spin states are emphasized, and techniques for calculating capture probabilities are developed. Capture probabilities are calculated for both the synchronous and the $1.5n$ resonance and are applied to Mercury and the moon. Similar techniques are used in a discussion of the Venusian spin, which may be commensurate with its synodic mean motion. A stability criterion is determined, and the possibility of capture into such a resonant state is considered. As a check on the validity of the averaged equations of motion the complete equations of motion were directly integrated in several cases, and the results were shown to agree with those derived from the averaged equations.

II. EQUATIONS OF MOTION AND STABILITY

Let us consider a planet whose spin axis is normal to its orbit plane. The principal moments of inertia are A, B, and C, where C is the moment about the spin axis. The orbit is taken to be a fixed ellipse specified by a semimajor axis a, eccentricity e, and instantaneous radius r. Position in the orbit is determined by the true anomaly f or the mean anomaly M. Other angles we use are displayed in Fig. 2. Line IF is an inertial line, whereas PS is the planet–sun center line. The angle between the planet's long axis and the center line is ψ. The angular position of the long axis relative to the inertial line IF is denoted by θ. From Fig. 2 we see that

$$\theta = f + \psi. \tag{1}$$

The equation of motion for θ is (Danby 1962),

$$C\ddot{\theta} + \tfrac{3}{2}(B-A)(GM_{\odot}/r^3)\sin 2\psi = 0, \tag{2}$$

where G is the gravitational constant and $M_{\odot}$ is the solar mass. In writing Eq. (1) we have specifically excluded the solar tidal torque as well as torques produced by other planets or satellites.

As it stands Eq. (2) is insoluble. Fortunately, in the cases of interest to us it may be reduced to a more tractable form. We are concerned with planetary spin rates having values near pn, where p is a half-integer and n is the orbital mean motion. It then becomes convenient to introduce a new angle γ such that

$$\gamma = \theta - pM. \tag{3}$$

Substituting Eqs. (1) and (3) into Eq. (2), we obtain

$$C\ddot{\gamma} + \frac{3}{2}\left(\frac{a}{r}\right)^3 n^2 \left(\frac{B-A}{C}\right)$$
$$\times \sin 2\gamma(\cos 2f \cos 2pM + \sin 2f \sin 2pM)$$
$$\times \cos 2\gamma(\cos 2f \sin 2pM - \sin 2f \cos 2pM) = 0, \tag{4}$$

If $\dot{\theta}$ is very close to pn and $(B-A)/C \ll 1$, γ will only change very slightly during each orbit. This suggests that we average Eq. (4) over an orbital period holding γ fixed. Using Cayley's tables (Cayley 1859), we find the averaged equation to be

$$C\ddot{\gamma} + \tfrac{3}{2}n^2(B-A)H(p,e)\sin 2\gamma = 0, \tag{5}$$

where $H(p,e)$ are power series in e. Several of the $H(p,e)$ are listed in Table I. One of their important properties is that the leading term in $H(p,e)$ is of order $e^{2|p-1|}$.

From Eq. (5), which is the ordinary pendulum equation, we observe that in the absence of tidal torques a planet could exhibit stable librations about spin rates of pn. If $H(p,e)$ is positive, the stable libration is about $\gamma = 0$, and the position of the planet's long axis at perihelion will oscillate about the solar direction. For $H(p,e)$ negative, the stable libration occurs about $\gamma = \frac{1}{2}\pi$, and now it is the planet's short axis which tends to point toward the sun at perihelion.

When tidal torques are present, the averaged (over an orbit period) tidal torque $\langle T\rangle$ must be added to the right-hand side of Eq. (5). If $\langle T\rangle$ is essentially constant (i.e., if $\langle T\rangle$ is independent of γ, $\dot{\gamma}$ and t), the stability condition for a resonant spin state is easily determined. For stability it is only necessary that the tidal torque not exceed the maximum possible restoring torque, which occurs at γ displaced by $\frac{1}{4}\pi$ from its equilibrium value. Thus the stability criterion becomes

$$|\langle T\rangle| < \tfrac{3}{2}(B-A)H(p,e)n^2. \tag{6}$$

An expression for the instantaneous tidal torque on a spinning planet is (MacDonald 1964)

$$T = -\frac{3k_2 G M_{\odot}^2 R^5}{2r^6}\sin 2\delta, \tag{7}$$

where k_2 is the tidal Love number and R the planet's

TABLE I. Coefficient of restoring torque to order e^4.

p	$H(p,e)$				
−1.0					$-e^4/24$
−0.5				$-e^3/48$	
0.5		$-e/2$		$+e^3/16$	
1.0	1		$-5e^2/2$		$+13e^4/16$
1.5		$7e/2$		$-123e^3/16$	
2.0			$17e^2/2$		$-115e^4/6$
2.5				$845e^3/48$	
3.0					$533e^4/16$

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mean radius. As MacDonald shows, $\delta \sim 1/Q$; this relation together with Eq. (6) may be used to express the sufficient stability condition for the pth resonance as

$$\frac{B-A}{C} > \frac{7.1\times10^{-8}}{Q|H(p,e)|}, \qquad (8)$$

for Mercury and

$$\frac{B-A}{C} > \frac{3.8\times10^{-7}}{Q|H(p,e)|}, \qquad (9)$$

for the moon. In arriving at Eqs. (8) and (9) we have assumed $k_2=0.05$ for Mercury and $k_2=0.02$ for the moon (MacDonald 1964). The Q's appropriate to Mercury and the moon almost certainly satisfy

$$10 < Q < 500$$

(Goldreich and Soter 1966). Even if $Q=10$ for Mercury, the large orbital eccentricity of 0.206 would permit a stable resonance to exist at $p=\frac{3}{2}$ for $(B-A)/C \gtrsim 10^{-8}$. In fact, if Mercury supported internal stress differences of the same order as those of the moon [which would imply $(B-A)/C \simeq 4\times10^{-5}$], many other resonant spins including all those with $-0.5 \leq p \leq 3.0$ would also be stable. For the moon $(B-A)/C \simeq 2\times10^{-4}$ (Jeffreys 1961). Using $Q=10$ and $e=0.0549$, we find that inequality (9) is satisfied when $0.5 \leq p \leq 2.5$ and possibly in other cases as well.

We have demonstrated the stability of several resonant spin states for both Mercury and the moon. We turn next to the question of capture into these states.

III. CAPTURE PROBABILITIES

We consider the rotation of a nonaxisymmetric planet as tidal friction slows its spin from an initial rate which is much larger than n. The orbit of the planet is assumed to be a fixed ellipse. Complications which arise when the orbit is varying are treated later.

As the planet's rotation slows, it will occasionally approach one of the resonant spin values. Near the pth resonance we may use the averaged equation of motion

$$C\ddot{\gamma} + \tfrac{3}{2}(B-A)n^2H(p,e)\sin 2\gamma = \langle T\rangle. \qquad (10)$$

If $\langle T\rangle$ is constant, the first integral of this equation is

$$\frac{C\dot{\gamma}^2}{2} - \tfrac{3}{4}(B-A)n^2H(p,e)\cos 2\gamma = \langle T\rangle\gamma + E_0, \qquad (11)$$

where $E=E_0+\langle T\rangle\gamma$ is referred to as the energy. For $E > \frac{3}{4}(B-A)n^2H(p,e)$, γ is an angle of rotation, whereas, for $E < \frac{3}{4}(B-A)n^2H(p,e)$, γ is an angle of libration. At the minimum energy, $E=-\frac{3}{4}(B-A)n^2H(p,e)$, the amplitude of libration vanishes.

In Fig. 3 we have plotted $\frac{1}{2}\dot{\gamma}^2$ vs γ using Eq. (11). The magnitude of $\langle T\rangle$ has been greatly exaggerated in order to better illustrate the principles involved. We have chosen γ initially as an angle of rotation with $\dot{\gamma}>0$. This corresponds to an initial planetary spin which is faster than the resonant one. From Fig. 3 we see that γ increases up to $\gamma_{\max}$, where $\dot{\gamma}$ vanishes. The motion then reverses with $\dot{\gamma}$ becoming negative and γ decreasing indefinitely. Since $\dot{\gamma}<0$ corresponds to a rotation rate less than the resonant one, we observe that the planet has successfully skirted the resonant spin state. Thus we may draw the general conclusion that if $\langle T\rangle$ is constant, the planet cannot be trapped at a spin rate which is commensurate with its orbital mean motion.

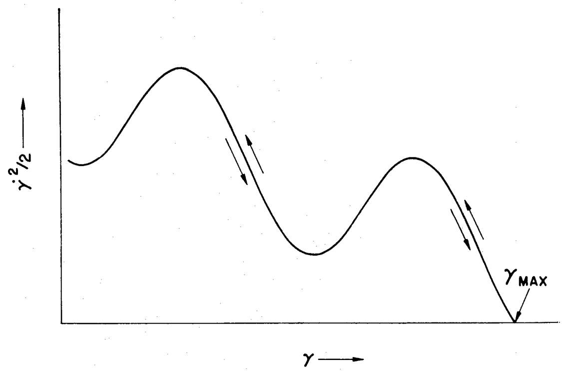

FIG. 3. Schematic diagram of $\frac{1}{2}\dot{\gamma}^2$ vs γ for $\langle T\rangle$ constant. No capture.

On the other hand, if the dependence of $\frac{1}{2}\dot{\gamma}^2$ on γ is not symmetric about $\dot{\gamma}=0$, but is as depicted in Fig. 4, the planet will be trapped into stable libration about the resonant spin value. What kind of tidal torque would give rise to this situation? Obviously, the torque must show a dependence on the sign of $\dot{\gamma}$. Near $\dot{\gamma}=0$, two simple torques of this form may be written as

$$\langle T\rangle = -K\left(V+\frac{\dot{\gamma}}{n}\right), \qquad (12a)$$

and

$$\langle T\rangle = -W - Z\,\mathrm{sign}\,\dot{\gamma}, \qquad (12b)$$

where K, V, W and Z are positive constants. We shall derive capture probabilities in terms of the torques given by Eqs. (12a) and (12b) and subsequently relate these torques to physical models of frictionally retarded tides.

We consider first $\langle T\rangle$ as given by Eq. (12a). Substituting this equation into Eq. (10) we arrive at

$$(d/dt)[\tfrac{1}{2}C\dot{\gamma}^2 - \tfrac{3}{4}(B-A)n^2H(p,e)\cos 2\gamma] = -K[V\dot{\gamma}+(\dot{\gamma}^2/n)], \qquad (13a)$$

or equivalently,

$$dE/dt = -K[V\dot{\gamma}+(\dot{\gamma}^2/n)]. \qquad (13b)$$

For $\dot{\gamma}>0$ the amount by which a minimum of $\frac{1}{2}\dot{\gamma}^2$ falls below the preceding one is denoted by ΔE. To illustrate this point in Fig. 4 the second minimum has been drawn (dashed) even though $\dot{\gamma}^2/2$ never reaches it. In actual cases, where the tidal torque is much smaller

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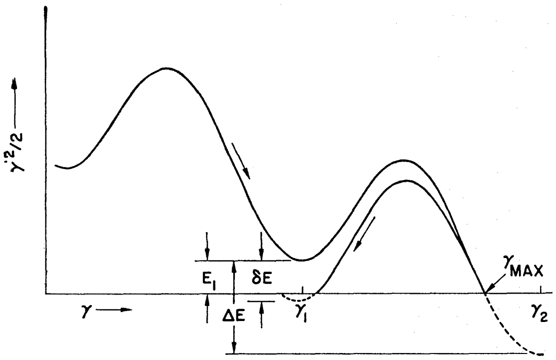

FIG. 4. Schematic diagram of $\dot{\gamma}^2/2$ vs γ for $\langle T\rangle = -K(V+\dot{\gamma}/n)$. Capture.

than depicted, the zero of $\frac{1}{2}\dot{\gamma}^2$ (at $\gamma=\gamma_{\max}$) occurs just before $\gamma=\gamma_2$. Thus, we may replace $\gamma_{\max}$ by γ_2 in the integrals which follow. From Fig. 4 and Eq. (13b), we have

$$\Delta E = -K\int_1^2 \left(V\dot{\gamma}+\frac{\dot{\gamma}^2}{n}\right)dt = -K\int_{\gamma_1}^{\gamma_2}\left(V+\frac{\dot{\gamma}}{n}\right)d\gamma, \quad (14a)$$

$$\Delta E' = -K\int_2^1 \left(V\dot{\gamma}+\frac{\dot{\gamma}^2}{n}\right)dt = -K\int_{\gamma_2}^{\gamma_1}\left(V+\frac{\dot{\gamma}}{n}\right)d\gamma, \quad (14b)$$

and

$$\delta E = \Delta E + \Delta E'. \quad (14c)$$

For weak tidal torques, the return path from γ_2 to γ_1 falls just slightly below the incoming path from γ_1 to γ_2. Hence

$$\delta E \simeq -\frac{2K}{n}\int_1^2 \dot{\gamma}^2 dt = -\frac{2K}{n}\int_{\gamma_1}^{\gamma_2}\dot{\gamma}d\gamma. \quad (15)$$

As we have drawn Fig. 4, $|\delta E| > E_1$, which leads to a second zero in $\frac{1}{2}\dot{\gamma}^2$, this time near $\gamma=\gamma_1$. Near γ_1, $\dot{\gamma}$ again reverses sign, and γ is now trapped into a libration between γ_1 and γ_2. The term in $\langle T\rangle$ which is proportional to $\dot{\gamma}$ will cause the amplitude of this libration to slowly decay. Thus the planet has become trapped in the resonant spin state. On the other hand, if $|\delta E| < E_1$ as displayed in Fig. 5, then the planet will escape the resonance and continue to despin. Therefore, unlike the case of constant $\langle T\rangle$, which never permits capture, when $\langle T\rangle$ contains a term proportional to $\dot{\gamma}$, trapping at a resonance will sometimes occur.

We may calculate the probability that a torque of the form given in Eq. (12a) will lead to capture. Since the values of E_1 are distributed with uniform probability between 0 and ΔE, the probability of capture is

$$P = \frac{\delta E}{\Delta E} = \frac{2}{1+\pi V\Big/\displaystyle\int_{\gamma_1}^{\gamma_2}\frac{\dot{\gamma}}{n}d\gamma}, \quad (16)$$

where we have used $\gamma_2-\gamma_1 \simeq \pi$. Of course, if $\delta E/\Delta E > 1$, $P=1$.

We again make use of the condition that the tidal torque is much smaller than the typical solar torque on the planet's permanent deformation from axial symmetry. Then, the solutions of equation (5), which does not include the tidal torque, represent excellent short-term approximations to the actual motions. Thus, the solution of this equation for $\dot{\gamma}$ may be substituted in the right-hand side of Eq. (16) in order to evaluate the capture probability. We are interested in the singular solution which separates the rotations from the librations [see discussion following Eq. (11)]. In this case we find

$$\dot{\gamma}/n = \{3[(B-A)/C]H(p,e)\}^{\frac{1}{2}}\cos\gamma, \quad (17)$$

hence

$$P = \frac{2}{1+\pi V/2[3(B-A)H(p,e)/C]^{\frac{1}{2}}}. \quad (18)$$

From the expression for P we see that a nonnegligible capture probability requires a small value for V, because $(B-A)/C$ is small for all planets and satellites. Thus a sizable fraction of the total tidal torque must be proportional to $\dot{\gamma}$ for capture to become likely. We emphasize that for $\langle T\rangle = -K(V+\dot{\gamma}/n)$ the capture probability is dependent on the value of $(B-A)/C$.

Let us repeat the arguments just given using $\langle T\rangle = -W - Z$ sign $\dot{\gamma}$ [Eq. (12b)]. Defining ΔE and $\Delta E'$ as before, we easily show that

$$\Delta E = -[W+Z)\pi, \quad (19a)$$

$$\Delta E' = +(W-Z)\pi, \quad (19b)$$

so that

$$\delta E = \Delta E + \Delta E' = -2\pi Z. \quad (19c)$$

Thus, the capture probability becomes

$$P = 2Z/(W+Z). \quad (20)$$

Unlike the previous case, this capture probability is *independent* of $(B-A)/C$.

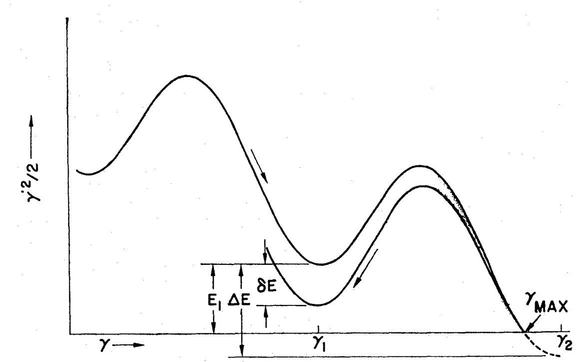

FIG. 5. Schematic diagram of $\frac{1}{2}\dot{\gamma}^2$ vs γ for $\langle T\rangle = -K(V+\dot{\gamma}/n)$. No capture.

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We shall employ expressions for the tidal torques which were developed by MacDonald (1964) and Darwin (1908). MacDonald assumes that the lunar tidal potential produces a second harmonic distortion of the earth. The effects of friction are modelled by delaying the time of high tide. Thus, unless the month and day are equal (and $e=0$), the tidal bulge is displaced by an angle δ from the position directly under the moon. An *ad hoc* feature of this model is that the angle δ incorporates the entire physics of the frictional process. In particular, for constant δ the time lag of the tide varies with the moon's position in orbit if $e\neq 0$ (see Kaula 1964).

In terms of his model MacDonald finds the instantaneous tidal torque to be that given by Eq. (7). We have used this expression for three different functional forms of δ. In all cases we have $\delta\ll 1$, so that $\sin 2\delta\sim 2\delta$. In Fig. 1 we have displayed the asymptotic tidal spin rates for the three types of δ.

In his development of the tidal disturbing function (Kaula 1964) follows the procedure first applied by Darwin (1908). The tidal potential at the planet is expanded in a Fourier time series. Each component of the series raises a tide in which the effects of dissipation are modelled by a phase lag, causing each component tide to lag behind the potential which raises it. The phase lag ϵ_i in each component tide is simply related to the planet's Q by

$$\sin\epsilon_i \simeq 1/Q. \tag{21}$$

If Q is independent of both tidal amplitude and frequency, as experiments at much higher frequencies on rocks suggest it might be (Knopoff 1964), then $|\epsilon_i|$ will be the same for all component tides. The sign of ϵ_i will be determined by the sign of the frequency of the component tide. Of course, we could also consider cases where $1/Q$ (and hence the ϵ_i) is proportional to frequency. However, qualitatively at least, these cases are similar to the ones obtained using MacDonald's torques [i.e., $\langle T\rangle = K(V+\dot\gamma/n)$] so we do not consider them.

It is easily shown (Kaula 1964) that the tidal torque on a planet whose spin axis is normal to its orbit plane is

$$\langle T\rangle = -K\sum_{p=-\infty}^{\infty}[H(p,e)]^2\,\mathrm{sgn}(\dot\theta-pn), \tag{22}$$

where $H(p,e)$ is given in Table I, K is a positive constant, and Q is assumed constant. For small e, $\langle T\rangle$ is negative for $\dot\theta>n$, and the tidal torque will drive the planet to synchronous rotation. For $e>0.235$, $\langle T\rangle$ changes sign from negative to positive at $\dot\theta=\frac{3}{2}n$. The final asymptotic spin rates are shown in Fig. 1 under the heading $Q=$ const.

Let us now proceed to calculate capture probabilities using first MacDonald's and then Darwin's forms for the tidal torque.

A. MacDonald's Torques

We shall show that these torques are well approximated by one of the two forms:

$$\langle T\rangle = \text{const or } \langle T\rangle = -K(V+\dot\gamma/n).$$

The simplest functional dependence occurs for δ proportional to tidal frequency. Then Eq. (7) assumes the form

$$T=-J(a/r)^6[(\dot\theta-\dot f)/n], \tag{23}$$

where J is a positive constant. Averaging T over an orbit, we find that $\langle T\rangle$ takes the form of Eq. (12a) with

$$V=p-\frac{(1+15e^2/2+45e^4/8+3e^6/16)}{(1+3e^2+3e^4/8)(1-e^2)^{\frac{3}{2}}}, \tag{24}$$

where Eq. (3) has been used. Use of Eq. (24) together with Eq. (18) yields the capture probability for the pth resonance when δ is proportional to tidal frequency.

If δ is dependent of both tidal frequency and amplitude, the expression for the averaged tidal torque becomes more cumbersome. In this case ψ, and hence the tidal torque, will change sign if Eq. (25) has a solution for $f=f_1$:

$$\cos f_1 = (\{[(p+(\dot\gamma/n)](1-e^2)^{\frac{3}{2}}\}^{\frac{1}{2}}-1)/e. \tag{25}$$

The reversals of the tidal torque will then occur at both $f=f_1$ and $f=2\pi-f_1$. In terms of f_1 we may write the averaged tidal torque in the form

$$\begin{aligned}\langle T\rangle = -D_1\{&(1+3e^2+3e^4/8)(\pi-2f_1)\\ &-(8e+6e^3)\sin f_1-(3e^2+e^4/2)\sin 2f_1\\ &-2e^3\sin 3f_1/3-e^4\sin 4f_1/16\},\end{aligned} \tag{26}$$

where D_1 is a positive constant. If $|\cos f_1|>1$, then there is no torque reversal, and f_1 must be set equal to zero in Eq. (26), which makes $\langle T\rangle$ independent of $\dot\gamma$. In general, to determine the transition probability for this type of tide, we must evaluate ΔE and $\Delta E'$ (see Figs. 4 and 5 for their definition) in terms of the more general expressions

$$\Delta E=\int_1^2\langle T\rangle\dot\gamma dt=\int_{\gamma_1}^{\gamma_2}\langle T\rangle d\gamma, \tag{27a}$$

$$\Delta E'=\int_2^1\langle T\rangle\dot\gamma dt=\int_{\gamma_2}^{\gamma_1}\langle T\rangle d\gamma. \tag{27b}$$

To express $\langle T\rangle$ as a function of $\dot\gamma$, and hence γ, we use Eqs. (25) and (17). The evaluation of these integrals may now be performed on a computer. However, if f_1 does not come too close to zero, we may derive approximate analytic expressions for ΔE and $\Delta E'$. Setting

$$f_1=f_0+\delta f, \tag{28}$$

where

$$\cos f_0=\{[p(1-e^2)^{\frac{3}{2}}]^{\frac{1}{2}}-1\}/e, \tag{29}$$

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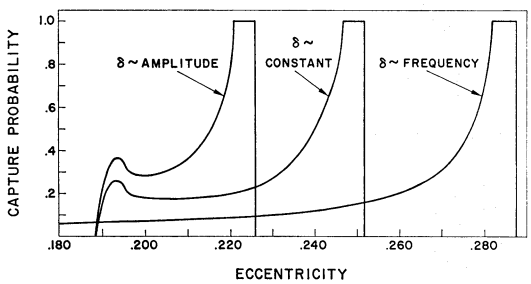

FIG. 6. Capture probability into $\frac{3}{2}$ resonance from faster spin for $(B-A)/C=10^{-4}$. MacDonald's torques.

we find

$$\delta f=-\frac{\dot{\gamma}(1-e^2)^{\frac{3}{2}}}{2\,\mathrm{ne}(p)^{\frac{4}{3}}\sin f_0}. \tag{30}$$

Substituting Eq. (30) into Eq. (26), we are able to put $\langle T\rangle$ in the form of Eq. (12a) with

$$V=-\frac{2e(p)^{\frac{1}{3}}\sin f_0}{(1-e^2)^{\frac{3}{2}}}\left[\frac{\langle T\rangle}{d\langle T\rangle/df_1}\right]_{f_1=f_0}, \tag{31}$$

Thus, even for constant δ, if the resonant angular velocity lies between the orbital angular velocities at perihelion and aphelion, the reversals of tidal torque will lead to a term proportional to $\dot{\gamma}$ in $\langle T\rangle$. Equation (31) may be used in Eqs. (14) to give approximate expressions for ΔE and $\Delta E'$.

For a linear amplitude dependence $\delta\propto(a/r)^3$. As for the case of constant δ, the averaged tidal torque may be written in terms of f_1 defined by Eq. (22). Now

$$\begin{aligned}\langle T\rangle=D_2\{&(1+21e^2/2+105e^4/8+63e^6/16)(2f_1-\pi)\\&+(14e+105e^2/2+105e^4/4+35e^6/16)\sin f_1\\&+(21e^2/2+35e^4/2+105e^6/32)\sin 2f_1\\&+(35e^3/6+105e^5/24+7e^7/32)\sin 3f_1\\&+(35e^4/16+105e^6/32)\sin 4f_1\\&+(21e^5/40+7e^7/160)\sin 5f_1\\&+(7e^6/96)\sin 6f_1+e^7/224\sin 7f_1\},\end{aligned} \tag{32}$$

where D_2 is a positive constant. An approximate expression for the tidal torque in the form given by Eq. (12a) may be derived. As in the case of constant δ,

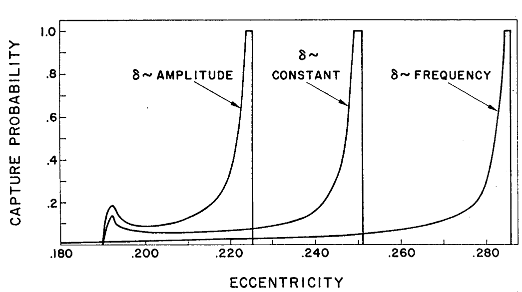

FIG. 7. Capture probability into $\frac{3}{2}$ resonance from faster spin for $(B-A)/C=10^{-5}$. MacDonald's torques.

the appropriate value of V, expressed in terms of the torque $\langle T\rangle$, is given by Eq. (31).

B. Darwin's Torques

These torques are of the form $\langle T\rangle=-W-Z\,\mathrm{sgn}\dot{\gamma}$ near each resonance [see Eqs. (12b) and (22)]. Thus, the probability that a despinning planet will be captured in the pth resonance as it despins is just

$$P=2[H(p,e)]^2\Big[\sum_{q=-\infty}^{\infty}|H(q,e)|^2\,\mathrm{sgn}(p+\eta-q)\Big]^{-1}, \tag{33}$$

where $0<\eta<\frac{1}{2}$. We emphasize once again the striking difference between the capture probabilities calculated here and those calculated from MacDonald's torques. Here we find P to be independent of $(B-A)/C$ with a nonzero capture probability at each resonance.

IV. APPLICATIONS TO MERCURY AND THE MOON

In this section we regard the orbits of Mercury and the moon as fixed ellipses. However, capture probabilities are described for wide ranges of eccentricity.

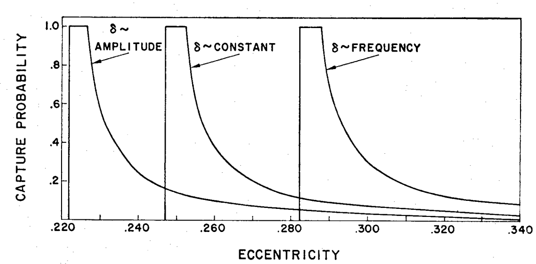

FIG. 8. Capture probability into $\frac{3}{2}$ resonance from slower spin for $(B-A)/C=10^{-4}$. MacDonald's torques.

A. Mercury

We shall discuss the possibilities of capturing Mercury at a resonance, first assuming tidal torques of MacDonald's form and then of Darwin's form with constant Q. The results, which are quite different in the two cases, are compared at the end of Sec. IVA.

1. MacDonald's Torques

For Mercury's orbit ($e=0.206$) the resonant spin values at $\dot{\theta}=n$, and $3n/2$ both lie between the orbital angular velocity at perihelion and aphelion. As long as $e<0.31$, the resonant spin state at $\dot{\theta}=2n$, as well as all states with $|p-1|>1$, will fall outside this range. The state with $p=\frac{3}{2}$ will remain in this range so long as $e>0.19$. It is important to distinguish the resonances which lie inside and outside this range, for we have seen in the previous section that when δ is independent of frequency, reversal of the tidal torque is a necessary condition for capture into a resonant spin state. Without reversal, even a δ proportional to frequency leads to only a very small capture probability. In

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Figs. 6–9 we have displayed capture probabilities for the $3n/2$ resonant spin as a function of eccentricity. Calculations involving δ proportional to frequency were made using Eqs. (18) and (24), whereas those involving δ constant and δ proportional to tidal amplitude were performed by integration of the complete tidal torques given by Eqs. (26) and (32), with $\dot{\gamma}$ determined as a function γ from Eq. (17). In the latter two cases, the approximate forms of $\langle T\rangle$, given by Eq. (31), were also used. The capture probabilities derived in this way were found to be in good agreement with the more exact values for $e>0.195$.

Figures 6 and 7 illustrate the capture probability into the $\dot{\theta}=\frac{3}{2}n$ state from an initially faster rotation. We observe that for a given type of δ, the capture probability is an increasing function of $(B-A)/C$. For all three types of δ, the capture probability drops to zero for e greater than some critical value. This may be explained by noting from Fig. 1 that for e larger than this critical value the asymptotic tidal spin is faster than $\frac{3}{2}n$. The vanishing of the capture probabilities for frequency independent δ's which occurs below another critical e is a consequence of the lack of

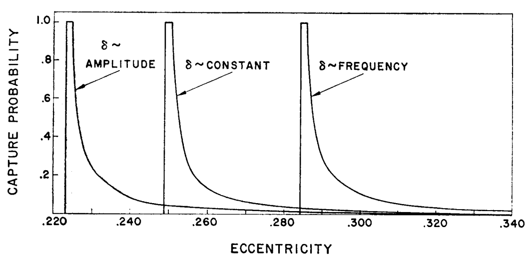

FIG. 9. Capture probability into $\frac{3}{2}$ resonance from slower spin for $(B-A)/C=10^{-4}$. MacDonald's torques.

reversal of the tidal torque below this value (when $\dot{\theta}=\frac{3}{2}n$). For δ proportional to frequency, this lower cutoff does not appear because now there is always a term in $\langle T\rangle$ proportional to $\dot{\gamma}$ [see Eq. (24)], whatever the value of e. However, even for $(B-A)/C=10^{-4}$, the capture probability at $e=0.190$ for δ proportional to frequency is only 0.08, making capture without torque reversal unlikely in all cases. The secondary maxima in the capture probabilities for δ independent of frequency, which occur near the lower cutoff, may at first seem surprising. However, they are a consequence of the vanishing of $d\dot{f}/df$ at perihelion, which implies that a small change in $\dot{\theta}$ will be reflected by an exaggerated change in f_1 [see Eq. (25)], and hence by a surprisingly large dependence of the tidal torque on $\dot{\gamma}$. The probability of capture is unity for a small range in e just below the upper cutoff, because near the resonance the average torque on the planet's permanent bulge is no longer zero. This average torque on the permanent bulge tends to bring the planet's rotation to the resonant value. It is just this effect

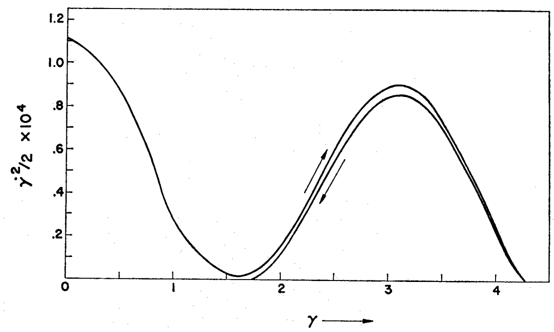

FIG. 10. Values of $\dot{\gamma}^2/2$ vs γ calculated from complete equation of motion with $(B-A)/C=10^{-4}$, $e=0.2$ and $\delta=$const. The value of n has been normalized to unity. Capture at $\frac{3}{2}$ resonance.

which allows satellites in orbits of low eccentricity to reach synchronous rotation (Goldreich 1966).

Figures 8 and 9 again illustrate capture probabilities into the $\frac{3}{2}n$ resonance, this time from initially slower spins. The principal features of these figures may all be understood by a simple adaptation of the discussion of Figs. 6 and 7.

In order to check the predictions of the averaged equations, we have performed several numerical integrations of the complete equation of motion. The equation used in the calculations was obtained by placing T in the form of Eq. (7) into the right-hand side of Eq. (2) and transforming the independent variable to f from t. Denoting differentiation with respect to f by a prime, this equation is

$$\psi''-2\left(\frac{r}{a}\right)\frac{e\sin f}{(1-e^2)}\psi'+\frac{3}{2}\left(\frac{B-A}{C}\right)\left(\frac{r}{a}\right)\frac{\sin 2\psi}{(1-e^2)}=2\left(\frac{r}{a}\right)\frac{e\sin f}{(1-e^2)}-D\left(\frac{a}{r}\right)^2\frac{\operatorname{sign}\psi'}{(1-e^2)},\quad(34)$$

where δ has been assumed constant and D is a positive constant. In all computer runs $(B-A)/C$ was set at the reasonable value of 10^{-4}. On the other hand, the unrealistically large value of 10^{-5} was used for D. The large value of D was chosen in order to keep the

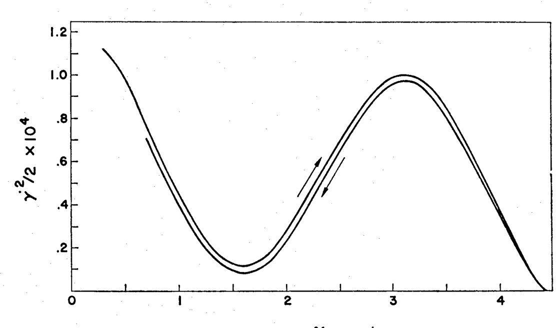

FIG. 11. Values of $\gamma^2/2$ vs γ calculated from complete equation of motion with $(B-A)/C=10^{-4}$, $e=0.2$ and $\delta=$const. The value of n has been normalized to unity. No capture at $\frac{3}{2}$ resonance.

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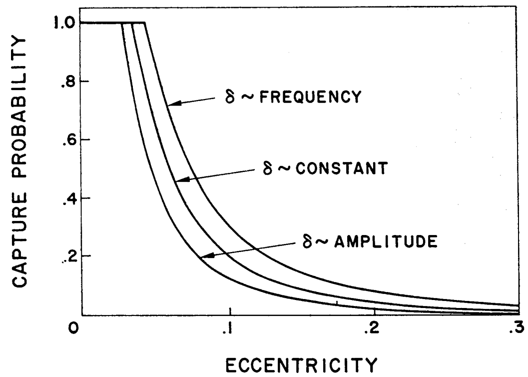

FIG. 12. Capture probability into synchronous resonance from retrograde spin for $(B-A)/C=10^{-4}$. Macdonald's torques.

computer time within reasonable bounds. Since our approximate criteria are independent of the magnitude of the tidal torque, as long as it is small compared to the torque on the planet's permanent bulge, this choice appears reasonable.

In Figs. 10 and 11 we have shown the results of two of six numerical integrations, all using $e=0.2$, the previously mentioned values of $(B-A)/C$ and D, and initial values of $\dot{\theta}/n$ slightly greater than 1.5. The six initial values of ψ were equally spaced between 0 and $\frac{1}{2}\pi$. The graphs of $\frac{1}{2}\dot{\gamma}^2$ vs γ are to be compared with the schematic drawing of Figs. 4 and 5. In Fig. 10 the planet is seen to have been trapped at the $\frac{3}{2}$ resonance, whereas in Fig. 11 it has escaped to a slower spin. The predicted capture probability of 0.16 (see Fig. 6) for these conditions agrees well with our record of one capture in six computer trials.

Unless Mercury's primordial spin was fortuitously close to its present value, the planet must have passed through either the $2n$ or n resonance. If Mercury passed the $2n$ resonance when $e \gtrsim 0.31$, there would have been no tidal torque reversal near this resonance, and if δ were independent of frequency, there would be zero probability for capture. Even if $(B-A)/C$ for Mercury is as large as 10^{-4}, the capture probability for δ proportional to frequency would be less than 3% at $e=0.21$. As capture in states with $p>2$ is even less likely, we could not expect Mercury to become trapped at any of the resonances higher than $p=\frac{3}{2}$. Figures 12 and 13 show the probability of capture at synchronous rotation from an initially retrograde rotation. Once again, these curves are easily understood in terms of the discussion of Figs. 6 and 7. In all cases, the capture probability is less than 10% at the present value of e. Therefore, an initially retrograde spin for Mercury could easily have been accelerated through the synchronous state to its present value.

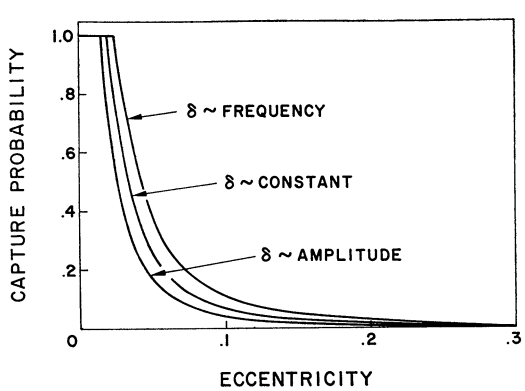

FIG. 13. Capture probability into synchronous resonance from retrograde spin for $(B-A)/C=10^{-5}$. MacDonald's torques.

2. *Darwin's Torques*

From Eqs. (33) and Table I, we may compute the probability that Mercury would have been captured at the $\frac{3}{2}$ resonance as it despun. This probability is displayed in Fig. 14. We note that for $e>0.235$, $P=1$. This reflects the change in sign of $\langle T\rangle$ as $\dot{\theta}$ crosses $\frac{3}{2}n$ for $e>0.235$. In this case, the asymptotic spin rate (as shown in Fig. 1) is also $\dot{\theta}=3n/2$, so that if $e>0.235$, a $\frac{3}{2}n$ spin would result even for an axially symmetric planet. In Figs. 15 and 16 we have drawn the capture probability at the $\frac{1}{2}n$ and n resonances from initially slower spins. These figures are obtained by setting $-\frac{1}{2}<\eta<0$ in Eq. (33) with $p=\frac{3}{2}$ and $p=1$. For $e=0.2$ the capture probabilities at the $p=2$ and $p=2.5$ resonances are 0.14 and 0.03, respectively.

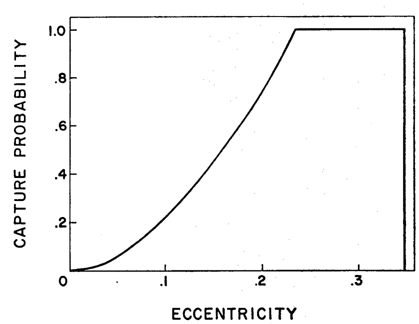

FIG. 14. Capture probability into 3/2 resonance from faster spin. Darwin's torque with constant Q.

On the basis of the observational data and the calculated asymptotic spin rates in Fig. 1, it seems probable that Mercury is trapped in the $p=\frac{3}{2}$ spin state. If any of MacDonald's torques or Darwin's torques with ϵ_i proportional to frequency are appropriate to Mercury, this would imply that $(B-A)/C$ for Mercury must be of order 10^{-4}. Even with this large value of $(B-A)/C$, the capture probability at the $\frac{3}{2}$ resonance is rather small $(\leq\frac{1}{3})$. On the other hand, if the appropriate tidal torque is that given by Darwin with $Q=$const, then the capture probability is >0.7 for $e=0.2$ and still >0.2 for $e=0.1$. Physically, Darwin's torque is much less arbitrary than MacDonald's. However, as we don't really understand the mechanism of frictional dissipation in planets, we prefer to reserve judgment on this matter until more information is available. We should mention that the crucial factor in Darwin's

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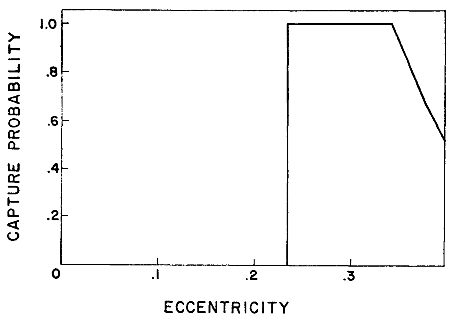

Fig. 15. Capture probability into 3/2 resonance from slower spin. Darwin's torque with constant Q.

torque with constant Q is just that the ϵ_i remain substantial for frequencies as low as $[(B-A)/C]^{\frac{1}{2}}n$. In fact, it is this property rather than the more highly specialized behavior with $Q=\text{const}$ that is essential for a substantial capture probability which is independent of $(B-A)/C$.

In any case, we have shown that Mercury was almost certainly initially spinning faster than $\frac{3}{2}n$ since the asymptotic spin rates are in general slower than this value for its orbital eccentricity. In addition, as Mercury despun the only resonance having a large capture probability (>0.15) was that with $p=\frac{3}{2}$.

B. Moon

A discussion of the moon's synchronous rotation has been given in a previous paper (Goldreich 1966). That investigation considered only the MacDonald-type torques and also neglected resonances with $p\neq 1$. We shall see that conclusions based on Darwin's torques are quite different from those reached previously.

1. MacDonald's Torques

Because the lunar orbital eccentricity is so low ($e\simeq 0.055$), the only resonance which is a likely candidate for the moon's final spin is synchronous rotation. However, the moon might still have gone to the asymptotic tidal state suggested by Peale and Gold (1965). Indeed, in a previous paper (Goldreich 1966) it was shown that the moon would not have achieved synchronous rotation (in its present orbit with its present figure) from an initially forward spin. It was concluded that the moon could have achieved synchronous rotation at some past date if $(B-A)/C$ were larger or e were smaller or its spin were initially retrograde. Of course, once the moon was trapped into synchronous rotation and no longer librating, it would remain trapped as long as the maximum torque on its permanent figure could exceed the tidal torque. In order to test the possibility for capture at synchronous rotation from an initially retrograde spin, seven numerical integrations of the complete equation of motion [Eq. (34)] were made using constant δ. In all cases, $(B-A)/C$ was given the derived value of 2×10^{-4} (Jeffreys 1961) while the tidal torque was set to $-T/C=2\times 10^{-5}$ sign $\dot{\psi}$, for reasons of economy previously described. In six of the seven trials the moon was trapped at synchronous rotation, whereas in the remaining case the faster asymptotic tidal rotation was achieved. Based on the averaged equations of motion we are able to compute the probability that synchronous rotation will arise from an initially retrograde spin. For $e=0.0549$ and $(B-A)/C=2\times 10^{-4}$, $P=0.71$, consistent with the meager statistics provided by the seven computer trials.

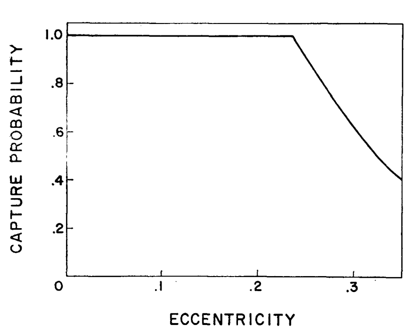

Fig. 16. Capture probability into synchronous resonance from retrograde spin. Darwin's torque with constant Q.

2. Darwin's Torques

As we have seen (Fig. 1) even an axially symmetric moon would reach synchronous rotation if Darwin's torques with $Q=\text{const}$ were acting on it and $e<0.235$. In this case, there would also have been a finite but small probability for capture at higher order resonances. For example, with $e=0.055$ the probability for capture at the $p=\frac{3}{2}$ and $p=2$ resonances becomes 0.07 and 0.005, respectively. We should mention that these probabilities are calculated on the assumption that the moon's spin axis is oriented normal to its orbit plane. The results of an investigation in progress suggest that the capture probability at the $p=2$ resonance might be considerably enhanced if the moon's spin axis possessed a large inclination to the orbital plane.

V. VENUS

If the sidereal rotation period of Venus is 243.16 days retrograde, the axis of Venus which points toward the earth at one inferior conjunction will point toward the earth at all subsequent inferior conjunctions. This value for the rotation period of Venus is compatible with the present range of observed values (Carpenter 1966; Goldstein 1965; Pettengill 1965). By a slight modification of the techniques developed in Sec. II we can determine a stability criterion for this synodic commensurability (Goldreich and Peale 1966b). The physical principle here is analogous to previously discussed coupling between a planet's spin and its own orbital motion. Here the inferior conjunction of Venus and the earth corresponds to perihelion passage for the planet, and the planetary orbit period is replaced by the Venusian synodic period.

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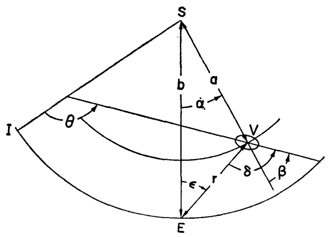

FIG. 17. Coordinates used in analysis of resonance of the second kind.

The orbits of Venus and the earth are approximated by concentric circles with the spin axis of Venus perpendicular to its orbit plane. It is also assumed that the axis about which the moment of inertia is minimal points approximately toward the earth at inferior conjunction. Figure 17 illustrates the angles and distances involved in the analysis. The points S, E, and V represent respectively the sun, the earth and Venus. The distances from the sun to Venus and the earth are denoted by ρ and b. The distance between the earth and Venus is r. The angle between the long axis (axis of minimal moment of inertia) and the inertial line SI is represented by θ, δ is the angle between the earth–Venus center line and the long axis of Venus, β is the angle between the sun–Venus line and the long axis, ϵ is the angle between the sun and Venus as seen from the earth and α is the difference in the heliocentric longitudes of the two planets. In the absence of solar tides, the equation of motion is (Danby 1962)

$$C\ddot{\theta}+\tfrac{3}{2}(B-A)\frac{GM_\oplus}{r^3}\sin 2\delta+\tfrac{3}{2}(B-A)\frac{GM_\odot}{\rho^3}\sin 2\beta=0, \quad (35)$$

where $M_\oplus$ is the earth's mass, and ρ is the sun–Venus separation.

In addition to exerting a torque on Venus about its center of mass, the earth also disturbs the orbit of Venus away from its assumed circular shape. Variations in the third term of Eq. (35) which arise from earth-produced perturbations of Venus's orbit are comparable to the earth's torque on the Venusian figure and hence must be included. We express the instantaneous position of Venus by

$$\rho=a+\Delta a, \quad \varphi=\varphi_0+\Delta\varphi, \quad (36)$$

where φ is the longitude of Venus in its orbit measured from inertial line SI, and $\Delta\varphi$ and Δa represent the periodic terms in φ and ρ due to the perturbation of Venus by the earth.

The condition for a commensurability of the type discussed here is that the average value of $\dot{\beta}$, which we denote by $\langle\dot{\beta}\rangle$, be a half-integral multiple of the average value of $\dot{\alpha}$. Hence, we have the following angular relationships:

$$\begin{aligned}&\beta=\theta-\varphi,\\ &\langle\dot{\beta}\rangle=p\langle\dot{\alpha}\rangle,\\ &\langle\dot{\theta}\rangle=n+p\langle\dot{\alpha}\rangle=\dot{\varphi}_0+p\langle\dot{\alpha}\rangle,\end{aligned} \quad (37)$$

where p is a half-integer and n is the mean orbital angular velocity of Venus. A retrograde rotation with a sidereal period of 243.16 days corresponds to $p=-5$. As we wish to investigate the stability of small changes in $\dot{\theta}$ about its mean value $\langle\dot{\theta}\rangle$, we introduce the angular velocity

$$\dot{\gamma}=\dot{\theta}-\langle\dot{\theta}\rangle, \quad (38)$$

as in Sec. II. If conjunction occurs along the inertial line SI at time $t=0$ (see Fig. 17), then from (37) and (38)

$$\begin{aligned}\beta&=\gamma+p\langle\alpha\rangle-\Delta\varphi,\\ \delta&=\gamma+(p+1)\langle\alpha\rangle+\epsilon.\end{aligned} \quad (39)$$

Substituting Eqs. (36), (38), and (39) into (35) and keeping only first-order terms in Δa and $\Delta\varphi$, we have

$$C\ddot{\gamma}+\tfrac{3}{2}(B-A)\frac{GM_\oplus}{b^3}\frac{b^3}{r^3}\sin 2[\gamma+(p+1)\alpha+\epsilon]+\tfrac{3}{2}(B-A)\frac{GM_\odot}{a^3}\Big[\sin 2(\gamma+p\langle\alpha\rangle)-\frac{3\Delta a}{a_0}\sin 2(\gamma+p\langle\alpha\rangle)-2\Delta\varphi\cos(2\gamma+2\langle\alpha\rangle)\Big]=0. \quad (40)$$

As the periodic part of Δa is symmetric about $\langle\alpha\rangle=0$ and that of $\Delta\varphi$ antisymmetric, they may be expressed as

$$\begin{aligned}\Delta a&=\sum_{i=1}^{\infty}\Delta a_i\cos i\alpha,\\ \Delta\varphi&=\sum_{i=1}^{\infty}\Delta\varphi_i\sin i\alpha,\end{aligned} \quad (41)$$

where any secular or constant terms are absorbed in redefined values for the sun–Venus distance and the mean motion of Venus. Substituting Eqs. (41) into Eq. (40) and averaging over a synodic period with γ held constant, we obtain the pendulum equation,

$$C\ddot{\gamma}+\tfrac{3}{2}(B-A)\frac{GM_\oplus}{b^3}[K_1(p)+K_2(p)]\sin 2\gamma=0, \quad (42)$$

where

$$K_1(p)=\frac{1}{2\pi}\int_0^{2\pi}\frac{b^3}{r^3}\cos 2\epsilon\cos 2(p+1)\alpha d\alpha-\frac{1}{2\pi}\int_0^{2\pi}\frac{b^3}{r^3}\sin 2\epsilon\sin 2(p+1)\alpha d\alpha \quad (43)$$

and

$$K_2(p)=\left(\frac{b}{a}\right)^3\frac{M_\odot}{M_\oplus}\left(\frac{-3\Delta a_{2p}}{2a}+\Delta\varphi_{2p}\right).$$

since

$$\cos 2\epsilon=1-\left(\frac{a}{b}\right)^2\left(\frac{b}{r}\right)^2+\left(\frac{a}{b}\right)^2\left(\frac{b}{r}\right)^2\cos 2\alpha$$

$$\sin 2\epsilon=\frac{a}{b}\sin\alpha\left\{\left[1-\left(\frac{a}{b}\right)^2\right]\left(\frac{b}{r}\right)^2+1\right\},$$

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TABLE II. Coefficients of restoring torque (Venus).

p	$K_1(p)$	$K_2(p)$	$K(p)=K_1(p)+K_2(p)$
−8.0	0.734	1.22×10^{-3}	.735
−7.5	0.921	1.88×10^{-3}	.923
−7.0	1.147	2.90×10^{-3}	1.150
−6.5	1.417	4.51×10^{-3}	1.424
−6.0	1.735	7.06×10^{-3}	1.742
−5.5	2.001	1.11×10^{-2}	2.012
−5.0	2.513	1.78×10^{-2}	2.531
−4.5	2.958	2.85×10^{-2}	2.987
−4.0	3.417	4.59×10^{-2}	3.463
−3.5	3.851	7.35×10^{-2}	3.924
−3.0	4.204	1.13×10^{-2}	4.317
−2.5	4.394	1.42×10^{-1}	4.536
−2.0	4.311	-8.10×10^{-2}	4.230
−1.5	3.831	−5.314	− 1.483
−1.0	2.857	1.843×10	21.29
−0.5	1.479	-1.240×10	−10.92
0.5	0.585	-2.622×10	−25.64
1.0	0.386	-7.805×10	−77.66
1.5	0.259	5.595×10	56.21
2.0	0.175	8.845	9.020
2.5	0.120	3.064	3.184
3.0	0.082	1.342	1.424
3.5	0.057	6.62×10^{-1}	.719
4.0	0.039	3.52×10^{-1}	.391
4.5	0.027	1.97×10^{-1}	.214
5.0	0.019	$1.14\times10\ ^{1}$	.133

$K_1(p)$ may be expressed in terms of the Laplace coefficients (Brouwer and Clemence 1961). However, as $K_1(p)$ was desired for several values of p, it was found more expedient to evaluate $K_1(p)$ by numerical integration on an IBM 7094 computer. The coefficients Δa_{2p} and $\Delta\varphi_{2p}$ may be evaluated from the conventional perturbation equations of celestial mechanics. Nevertheless, since we consider coplanar circular orbits, the series involved are just Fourier series, and it is then more convenient to evaluate Δa_{2p} and $\Delta\varphi_{2p}$ by directly Fourier analyzing the equation of motion of Venus, including perturbations by the earth. The earth's orbit is constrained to be a circle throughout the analysis because perturbations in the earth's position would only lead to second-order terms in the restoring torque. In a straightforward manner we obtain

$$K_2(p)=\left(\frac{b}{a}\right)\frac{1}{(S^2-1)}\left\{\left(\frac{1}{2}-\frac{1}{S}-\frac{3}{S^2}\right)\right.$$
$$\times[J(2p+1)-H(2p+1)]+\left(\frac{5}{2}+\frac{5}{S}+\frac{3}{S^2}\right)$$
$$\times[J(2p-1)+H(2p-1)]\mp\delta_{\pm1}{}^{2p}\left(1+\frac{3}{S}+\frac{3}{S^2}\right)$$
$$\left.-\delta_{\pm1}{}^{2p}\left(\frac{3}{2}+\frac{1}{2S}\right)\right\},\tag{44}$$

where

$$J(x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{b^2}{r^2}\cos\epsilon\,\cos x\alpha d\alpha,$$

$$H(x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{b^2}{r^2}\sin\epsilon\,\sin x\alpha d\alpha,$$

$S=2p\langle\dot{\alpha}\rangle/n$, and $\delta_j{}^i$ is the Kronecker delta. We evaluate $K_2(p)$ by numerical integration on an IBM 7094 computer. Values of $K_1(p)$, $K_2(p)$ and $K(p)=K_1(p)+K_2(p)$ are listed in Table II for several values of p.

From Table II we see that for values of $|p|\leq1.5$ the indirect portion of the restoring torque, which arises from perturbations of the orbit, dominates the direct gravitational torque of the earth. However, at the crucial resonance with $p=-5$ the direct torque is more than 100 times larger. The rotation rates which are commensurate with the Venusian synodic motion are seen to be stable in the absence of any external torques other than those due to Venus's permanent deformation from axial symmetry. The pth resonance will remain stable if any additional torque $\langle T\rangle$ about Venus's center of mass satisfies

$$|\langle T\rangle|<\tfrac{3}{2}K(p)(B-A)(GM_\oplus/b^3).\tag{45}$$

A solar tidal torque must be acting on Venus. Using Eq. (7) we see that if this were the only significant additional torque, on Venus the criterion for stability would become

$$\frac{B-A}{C}>\frac{4\times10^{-2}}{K(p)Q}=\frac{1.6\times10^{-2}}{Q},\tag{46}$$

for $p=-5$. Estimates between 10 and 20 for the earth's tidal effective Q are derived from observation of the secular acceleration of the moon (MacDonald 1964). These low values may be primarily due to dissipation in shallow seas (Miller 1964), and therefore may not be applicable to Venus. Estimates of the Q for the earth's mantle are on the order of several hundred (Knopoff 1964). A stable resonance with a Q of this order for Venus would require a value of $(B-A)/C$ comparable to that of the moon. It is not at all certain that Venus could maintain such a deformation since the larger surface gravity of Venus would imply internal stresses about 25 times greater than those supported by the moon (Kaula 1963).

If Venus had a primordial forward spin, the solar tides could not account for the present retrograde rotation. This has led some authors to propose the existence of other torques on the planet, notably of atmospheric origin (Gold, private communication 1964; MacDonald 1964). As the mass of the Venusian atmosphere is thought to be comparable to that of the earth's oceans, such proposals seem quite reasonable. However, if the atmosphere is capable of pushing Venus through the otherwise stable synchronous state of rotation, present control of the rotation of Venus by the earth would be hard to understand.

Elsewhere in this paper we have shown that there must be a substantial term in the averaged tidal torque which is dependent on γ in order that there be a significant probability of capture of a planet into a spin state which is resonant with its orbital mean motion. A similar requirement applies to the possible synodic resonance of Venus's spin. As yet we see no physical basis for the existence of such a term.

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Resonances of the Venusian spin with its motion relative to Jupiter are also possible. However, the only resonance of this kind which is compatible with the observed limits on the rotation period has less than $\frac{1}{10}$ the strength of the $p=-5$ resonance with the earth.

VI. ORBITAL VARIATIONS AND SHORT-PERIOD FLUCTUATIONS

Up to now, with the exception of some machine integrations of the complete equation of motion [Eq. (34)], we have based all our conclusions on the averaged equations of motion. Thus we have neglected those short-period variations in the planetary rotation rate which average to zero over a single orbit period. Furthermore, we have confined our investigation to fixed elliptical orbits and have yet to discuss how the naturally occurring variations in the orbit parameters affect our conclusions.

We consider the short-period variations first. In Eq. (2), the term of largest amplitude in the solar torque which averages to zero is of order $(B-A)n^2 \times \sin 2(p-1)nt$ (for $p\neq 1$). This fluctuating torque gives rise to short-period variations in $\dot{\gamma}$ with amplitude $\simeq n(B-A)/C$. Near the singular state separating the rotations from the librations $\dot{\gamma}\simeq[(B-A)H(p,e)/C]^{\frac{1}{2}}n$. Thus the short-period fluctuations in $\dot{\gamma}^2$ are of approximate magnitude $\Delta\dot{\gamma}^2\simeq[(B-A)/C]^{\frac{3}{2}}[H(p,e)]^{\frac{1}{2}}n^2$, whereas the ratio $\Delta\dot{\gamma}^2/\dot{\gamma}^2\simeq[(B-A)H(p,e)/C]^{\frac{1}{2}}$. For the computer results displayed in Figs. 10 and 11, $(B-A)/C\simeq 10^{-4}$ and $H(p,e)\simeq 0.7$, so that $\Delta\dot{\gamma}^2/\dot{\gamma}^2\simeq 10^{-2}$. The small value of $\Delta\dot{\gamma}^2/\dot{\gamma}^2$ explains the absence of short-period wiggles in Figs. 10 and 11.

Mutual perturbations between planets produce slow changes in their orbital parameters. Typical of these changes are the variations in eccentricity and perihelion position of Mercury's orbit which have been reported by Brouwer and Clemence (1961). The rotation of the apsidal line and the oscillation of e (predicted to range from 0.11 to 0.24) take place in about 220 000 yr. Variations of the semimajor axis, node and inclination must also occur on similar time scales. Do these perturbations affect the probability of capture into resonant spin states? We show that in essence they do not.

For simplicity, we restrict our considerations to the perturbations in orbital eccentricity. This is sufficient to illustrate the salient features without unnecessarily complicating our analysis. Because the large variation in orbital eccentricity occurs over many libration or rotation periods of the angle γ, the derivation of the averaged equation of motion (5) proceeds exactly as before. The only difference that arises is in its interpretation, with $H(p,e)$ now being a slowly varying function of time which must be evaluated at the appropriate value of e. The most important consequence arising from the variation of e in Eq. (5) is that the energy E is no longer conserved. However, since the variation in e is slow (compared to a period of motion of γ) the action

$$J=C\oint\dot{\gamma}d\gamma \tag{47}$$

is adiabatically conserved in the absence of tidal torques (Landau and Lifshitz 1960). The rate of change in E due to the varying eccentricity alone (i.e., $J=$const) is derived from

$$\frac{dJ}{dt}=\frac{\partial J}{\partial e}\frac{de}{dt}+\frac{\partial J}{\partial E}\frac{dE}{dt}\bigg|_J=0 \tag{48}$$

and

$$P=\partial J/\partial E,$$

where P is the period of γ's motion. Thus

$$\frac{dE}{dt}\bigg|_J=-\frac{1}{P}\frac{\partial J}{\partial e}\frac{de}{dt}. \tag{49}$$

On the other hand, the rate of change in energy due to the tidal torque alone (i.e., at constant e) is given by

$$dE/dt|_e=\langle T\rangle\dot{\gamma}. \tag{50}$$

Previously, this was the only means by which the energy could change, and the critical energy which separates the rotations from the librations was constant. Now the critical energy varies as a function of eccentricity, which complicates the calculation of capture probabilities.

In general, in order to follow the tidal history of a despinning planet in an orbit of varying eccentricity we must calculate the behavior of the energy as it is changed by both tidal forces and the variation of eccentricity. At each stage the energy must be compared with the instantaneous value of the critical energy (see Sec. II). When the energy approaches the critical value, we must calculate the probability that either libration or reversed rotation ($\dot{\gamma}$ changing sign) will ensue. This probability may be calculated by the method outlined in Sec. III, except now changes in $\frac{1}{2}\dot{\gamma}^2$ brought about by varying eccentricity must also be included. It is now possible for many temporary captures and escapes from libration to occur before the planet ultimately escapes or is captured.

Monte Carlo calculations of this general sort could be carried out on a large computer. However, because of the impossibility of accurately calculating the past variations of Mercury's orbit, these calculations would not appear to be very meaningful. Instead, in order to assess more simply the effects of varying eccentricity on capture probabilities, we consider two limiting cases.

In the first, the rate of change of eccentricity is assumed to be so slow that near resonance $[dE/dt|_e]\gg[dE/dt|_J]$, where the square brackets refer to averages over a period of motion of γ. In this case, the calculations of capture probability are just as before with $de/dt=0$. Furthermore, the critical value of energy is reached only once, with the subsequent capture or escape being permanent. In this limit, since we cannot know the value of eccentricity at which the

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critical energy will be reached, the over-all capture probability is the time average of the capture probability considered as a function of eccentricity.

In the second case, we assume de/dt sufficiently large so that $[dE/dt|_J] \gg [dE/dt|_e]$. Then the eccentricity could go through many oscillations while $\dot{\gamma}$ is near zero. However, we must emphasize again that the eccentricity will vary slowly compared to the variation in γ. In this limit, the planet may be temporarily captured into libration and subsequently released many times (at either faster or slower spins than the resonant one) before permanent capture or escape is determined. In order to avoid dealing with the rapid variations of energy due to the oscillating eccentricity it is expedient to work in terms of the adiabatic action invariant J. Unlike the energy the action is altered only by tidal friction. This variation is evaluated from

$$\frac{dJ}{dt} = \frac{\partial J}{\partial E}\bigg|_e \frac{dE}{dt} + \frac{\partial J}{\partial e}\bigg|_E \frac{de}{dt} = P\langle T\rangle\dot{\gamma}, \tag{51}$$

where we have made use of the adiabatic invariance of J together with Eq. (50) for $dE/dt|_J$. The critical value of J separating the rotations from the librations obtained from Eqs. (17) and (47) is

$$J_{\rm crit} = 4C\{3[(B-A)/C]H(p,e)\}^{\frac{1}{2}}n. \tag{52}$$

Let the extreme values of $J_{\rm crit}$ corresponding to the extreme values of e be denoted by $J_{\rm max}$ and $J_{\rm min}$. Soon after the tidal torque brings J to a value less than $J_{\rm max}$, $J_{\rm crit}$ will swing up to its maximum value and temporarily exceed J. During this time γ will librate. As $J_{\rm crit}$ then decreases it will again drop below J, at which time γ will again become an angle of rotation. In the limit of rapid variation of e it is equally likely that this rotation will be either faster or slower than the resonant value. If $\dot{\gamma}>0$, J will decrease, whereas if $\dot{\gamma}<0$, it will increase. If the latter occurs, J may increase above $J_{\rm max}$ before $J_{\rm crit}$ can exceed J, allowing the planet to permanently escape to a spin below the resonant value. Otherwise, $J_{\rm crit}$ will again rise above J, and the planet's spin will librate about the resonant value again. This pattern of capture and escape will continue until the planet has either permanently escaped the resonance ($J>J_{\rm max}$ and $\dot{\gamma}<0$) or until it has become permanently trapped ($J<J_{\rm min}$). What now determines the probability of capture?

Suppose $\langle T\rangle$ is constant. From Eq. (51) we then observe that the average value of dJ/dt over a libration period will vanish. Furthermore, when γ is an angle of rotation, the sign but not the magnitude of dJ/dt will depend on the sign of $\dot{\gamma}$. Thus, from the time J first drops below $J_{\rm max}$ it will undergo a random walk with steps of approximately equal size in both directions (from a given value of J). One step is the variation of J during a complete oscillation of e (and hence of $J_{\rm crit}$). If J nears $J_{\rm min}$, the fraction of the oscillation period of e over which $J>J_{\rm crit}$, and hence over which γ is an angle of rotation, will approach zero. Hence, J cannot reach $J_{\rm min}$ since the step size vanishes for J approaching $J_{\rm min}$. Thus, the planet must eventually escape to a slower than resonant spin, a return to $J>J_{\rm max}$ (however, now with $\dot{\gamma}<0$) being the certain outcome of this random walk. This result generalizes our previous one that for $\langle T\rangle$ constant, capture into a resonant spin state is impossible (see Sec. III). The arguments given here are easily extended to include all possible ratios of $[dE/dt|_e]$ to $[dE/dt|_J]$.

If $\langle T\rangle$ is not constant, but instead is of the form given by Eq. (12), J will decrease when γ is an angle of libration. Furthermore, the step size no longer approaches zero if J nears $J_{\rm min}$, since J near $J_{\rm min}$ implies that γ is almost always an angle of libration, and J decreases during libration. In fact, it is clear that for J sufficiently close to $J_{\rm min}$, all steps are towards lower J. Therefore, a term in $\langle T\rangle$ dependent on $\dot{\gamma}$ may lead to capture.

In order to assess the effect of varying orbital parameters on capture probabilities, we have performed several thousand Monte Carlo trials on a large computer. These were all performed in this limit of fast variation of orbital parameters by the method outlined in the previous paragraphs. As a result, we are able to state with some confidence that the over-all capture probability in an orbit of varying eccentricity is intermediate between the largest and smallest values of the capture probability (calculated for fixed orbits) over the range of eccentricity.

As previously mentioned, the value of eccentricity of Mercury's orbit has been calculated to oscillate between 0.11 and 0.24 with a period of 220,000 yr. Taking this variation in e and averaging Eqs. (49) and (50) over a period of γ's motion near the $p=\frac{3}{2}$ resonance, we arrive at

$$\left[\frac{dE}{dt}\bigg|_e\right] \bigg/ \left[\frac{dE}{dt}\bigg|_J\right] \lesssim \frac{15}{Q[(B-A)/C]^{\frac{1}{2}}}. \tag{53}$$

In deriving inequality (53) we have neglected reversals in $\langle T\rangle$, thus obtaining an upper bound on the ratio. If the asymptotic tidal state is near the resonance, the averaged tidal torque $\langle T\rangle$ is reduced by a factor of order $[(B-A)/C]^{\frac{1}{2}}$. This reduces the right-hand side of inequality (49) to about $15/Q$, or to about 0.1 to 1.0 for the Q values believed appropriate to Mercury. Thus for Mercury, the value of de/dt lies between the two limiting cases discussed in this section.

VII. CONCLUDING REMARKS

We find that there are many possible stable spin states in the solar torque which require only a small permanent deformation of the planetary figure. If MacDonald-type torques are applicable, larger deformations are needed to make capture likely. For Mercury, capture into the $\frac{3}{2}$ resonance has a significant probability only if $(B-A)/C \gtrsim 10^{-5}$. On the other hand, if Darwin-type torques with constant Q are applicable

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the capture probability at the $\frac{3}{2}$ resonance is greater than 0.5, independent of Mercury's $(B-A)/C$. These conclusions appear to be substantially unaltered by variations of the planetary orbit as long as these variations are slow compared with the libration period about the resonant spin.

A value of $(B-A)/C \gtrsim 10^{-4}$ appears to be the minimum required to stabilize the possible synodic resonance of Venus's spin. In addition to implying unusually large stress differences in the planet's interior, capture into this resonance appears to be difficult.

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Lunar Disturbing Function

D. Barton
St. John's College, Cambridge, England
(Received 1 April 1966)

This paper describes how, using a new computing technique, it has been possible to expand the lunar disturbing function in terms of the elliptic elements and the mean anomalies to the tenth order of small quantities. The result is identical with Delaunay's development to the eighth order. The ninth- and tenth-order terms are new.

The problem of a completely literal solution to the "main problem of the lunar theory" has received little attention since Delaunay's development to the eighth order of small quantities in 1860 (Delaunay 1860). It is evident that to improve this development in a reasonably short time it will be necessary to enlist the aid of a computer to perform the detailed manipulation of the multiple Fourier series that arise during the course of the work. The author has written a set of programs for the Titan computer in Cambridge to carry out the literal algebraical and trigonometrical manipulation required by this problem, and these programs have been applied to the derivation of the disturbing function and to performing the contact transformations of the type used by Delaunay and of the type with determining function as described by Brouwer and Clemence (1961). In this paper we discuss the derivation of the disturbing function.

Let us define S to be the ring of elements s, where s is a multiple Fourier series in up to six variables (called harmonic variables) with coefficients drawn from the ring of polynomials in eight variables (called polynomial variables) over the rational field. Let us define addition in S in the usual manner, and multiplication to be the normal product reduced to its linear form in the trigonometrical functions. We refer to this linear form as the standard form. A study of Delaunay's work shows that the manipulations required for the lunar theory may be carried out upon a computer provided that the computer has facilities to perform the following operations upon members of S.

1. Addition
2. Subtraction
3. Negation
4. Multiplication by a rational number
5. Selection of particular terms from a member of S

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