The oscillations of rapidly rotating stars

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ABSTRACT

We review the main aspects of modelling the oscillations of rotating stars. We first review the recent progress concerning the role of the Coriolis force when its amplitude is such that perturbative approaches are inappropriate. We insist on the qualitative changes which are brought on by this force and which are due to the ill-posed nature of the underlying mathematical problem. We then discuss the problems raised by the existence of the centrifugal acceleration and show that although of second order it should affect quite strongly the spectrum of acoustic modes of rapidly rotating stars. We then draw some conclusions on the route to follow in order to achieve the modelling of the oscillations of rotating stars.

1. Introduction

In this contribution we review the main aspects of modelling the oscillations of rotating stars. We first review the recent progress concerning the role of the Coriolis force when its amplitude is such that perturbative approaches are inappropriate. We insist on the qualitative changes which are brought on by this force and which are due to the ill-posed nature of the underlying mathematical problem (section 2). We then discuss the problems raised by the existence of the centrifugal acceleration and show that although of second order it should affect quite strongly the spectrum of acoustic modes of rapidly rotating stars (section 3). We then draw some conclusions on the route to follow in order to achieve the modelling of the oscillations of rotating stars.

2. Spectrum of oscillations with Coriolis force

The role of the Coriolis force on the oscillations of a star is twofold: it perturbs already existing modes (like acoustic, gravity or Alfven modes) and introduces its own modes, namely inertial modes, for which it is the restoring force.

Inertial modes occupy the frequency band $[0, 2\Omega]$ where Ω is the mean angular velocity of the star. As is well known, gravity modes occupy the frequency band $[0, N_{\max}]$ where $N_{\max}$ is the maximum Brunt-Väisälä frequency. Hence, gravity modes and inertial modes share a common frequency band $[0, \min(2\Omega, N_{\max})]$ where they form the spectrum of gravito-inertial modes.

In radiative zones of stars, $N_{\max} > 2\Omega$ and we usually have to deal with gravity modes perturbed by rotation and gravito-inertial modes while in convective zones $N_{\max} \sim 0$ implies that we are left with pure inertial modes. Let us add that acoustic modes being in the high frequency range are usually only slightly perturbed by the Coriolis force.

Pure inertial modes being constrained to convective zones, we should not expect that they reach large amplitudes as their excitation is stochastic. On the other hand gravity modes excited by the κ-mechanism can reach large amplitudes and, if their frequency is close to 2Ω, they will be strongly perturbed by rotation; an example of such a case may be found with γ-Doradus stars. We shall now discuss the effects of this strong coupling.

One of these effects is of mathematical origin and thus, to fully appreciate its origin, we shall first concentrate on the simple case of an incompressible inviscid fluid contained in a spherical shell, which mimics the convective or radiative zone of a star. Such a configuration only possesses inertial modes but this is sufficient for our purpose. The pressure perturbations associated with inertial modes verify the Poincaré equation (see Greenspan 1969), namely

$$\Delta p - \left(\frac{2\Omega}{\omega}\right)^2 \frac{\partial^2 p}{\partial z^2} = 0 \tag{1}$$

where ω is the frequency of the mode. It can be shown that $|\omega| \leq 2\Omega$ (Greenspan 1969), which makes Poincaré's equation of hyperbolic type, *i.e.* of the form:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \beta^2 \frac{\partial^2 p}{\partial z^2} = 0$$

However, fluid motions verify the boundary condition $\vec{v} \cdot \vec{n} = 0$ where $\vec{n}$ is a vector normal to the boundary; this boundary condition can also be expressed as a boundary condition on p:

$$-(\omega/2\Omega)^2 \vec{n} \cdot \vec{\nabla} p + i(\omega/2\Omega)(\vec{e}_z \times \vec{n}) \cdot \vec{\nabla} p + (\vec{e}_z \cdot \vec{n})(\vec{e}_z \cdot \vec{\nabla} p) = 0 \tag{2}$$

What is shown here is that the problem is *mathematically ill-posed* since a hyperbolic equation requires initial conditions rather than boundary conditions. This mathematical property has major consequences on the form of eigenmodes of a rotating fluid: in general there are no smooth solutions and therefore no eigenmode exists for an inviscid fluid. In other words, if we consider the (damped) eigenmodes of the associated viscous problem, their kinetic energy tends to infinity as viscosity decreases to zero.

Such a property is not specific to inertial modes and is shared also by gravity modes. Hence, these latter modes also suffer from these singularities in general (see Maas et al. 1997, Rieutord & Noui 1999) but in the case of a nonrotating star, the spherical symmetry of the configuration permits the separation of variables and thus removes the singular solutions; however, as soon as separability disappears singularities come back.

We shall now briefly describe this general situation when variables cannot be separated and refer the reader to Dintrans et al. (1999) and Rieutord et al. (2001) for more details.

As in all hyperbolic systems, characteristic lines[1] play a central role on the shape of solutions. In the standard wave equation, they materialize the propagation of energy (backward or forward

[1] They are the trace of characteristic surfaces in a meridional plane; these surfaces are axisymmetric and therefore fully characterized by their trace on a meridional plane.

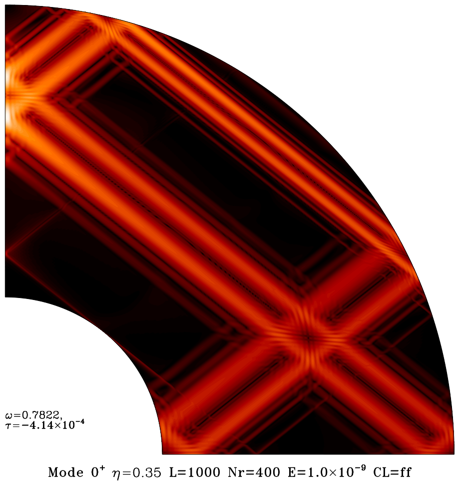

Fig. 1.— The kinetic energy distribution in a meridional plane of an inertial mode in a spherical shell associated with an equatorial attractor. The mode is axisymmetric with equatorial symmetry. Stress-free boundary conditions have been used on both shells; this solution was computed with an Ekman number of 10^{-9} and required 1000 spherical harmonics and 400 radial grid points (Gauss-Lobatto). The ratio of the inner radius to the outer radius is $\eta = 0.35$. $\omega = 0.7822$ and the damping rate $\tau = -4.14 \times 10^{-4}$ are given in units of 2Ω.

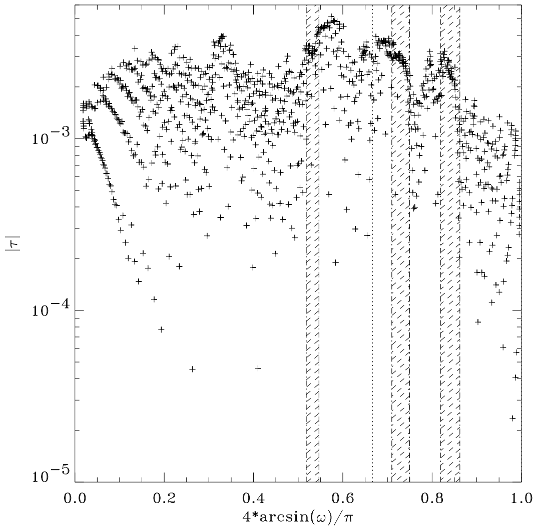

Fig. 2.— Distribution of eigen values in the complex plane for inertial modes of a spherical shell: x-axis is the arc sine of the frequency (normalized) and the y-axis (corresponding to the real axis) is the damping rate. The hatched frequency bands correspond to attractors; in stars such bands are much narrower as shown in Dintrans and Rieutord (2000); note that modes in these bands are much more damped than modes outside these bands.

as there are always two characteristics crossing at a given point for a second order equation). In the case of stars, the system is in fact of mixed type for all frequencies in $[0, \sqrt{N^2_{\rm max} + (2\Omega)^2}\,]$ which means that at a given point of the star the operator governing eigenmodes is either elliptic or hyperbolic. The surfaces separating elliptic and hyperbolic regions are called critical surfaces. Modes "live" inside the hyperbolic domain and see as a boundary either the true surface of the star or a critical surface. The structure of such modes, which can be inertial, gravity or gravito-inertial, is ruled by the trajectories of the characteristics inside this hyperbolic domain.

The trajectories of characteristics inside a hyperbolic domain may be viewed as that of a ball inside a billiard. However, unlike the ball which can move in any direction at each point, characteristics can take only two directions; thus, pushing the comparison, we may say that characteristics have much less freedom and that their phase space is much smaller. In fact, the phase space is so small that no chaos is allowed and trajectories of characteristics are either periodic, either wandering indefinitly (*i.e.* ergodic) or converging towards an attractor (a point or a periodic orbit). This latter behaviour is illustrated in Fig. 1 in the simple case of an inertial mode; other examples in more complex configurations may be found in Dintrans et al. (1999).

The focusing of characteristics towards attractors has immediate consequences on the shape of eigenmodes: they are singular. Their kinetic energy is not bounded and, strictly speaking, they do not exist. However, in physical systems, diffusion is always present even if very small. In this case the singularity is regularized but still exists in the form of shear layers as shown in Fig. 1 (or in Dintrans & Rieutord 2000 for a star).

One important property of singular modes is that they are generic : they are the rule while the regular modes are the exception. As a consequence the eigenvalue spectrum of these modes is very different from the eigenvalue spectrum of modes associated with an elliptic operator like acoustic modes. Indeed, shear layers associated with the singularities increase very much the damping rates. If E is a dimensionless number measuring viscosity or heat diffusion (in stars $E \ll 1$), the damping rate scales $\sqrt{E}$ instead of E as for a regular mode.

We may therefore expect that low-frequency modes will be difficult to excite in rotating stars compared to nonrotating stars. Quite surprisingly, this effect is not very strong: indeed, even though attractors are ubiquitous (in frequency space), they are mostly of very long period which implies that shear layers appear at extremely low diffusion usually well below the actual values in stars. Hence, the very small diffusion of stellar interiors is sufficient to smooth out most of the attractors and only a few of them survive in narrow frequency bands of the spectrum as illustrated by Fig. 2.

To conclude this section, let us emphasize that the Coriolis force deeply modifies the shape of eigenmodes in the range $\omega \lesssim 2\Omega$ and that diffusion (viscosity or thermal conductivity) is a necessary ingredient to properly compute these modes. Adiabatic calculations may still give 'good looking' results, especially if the numerical method is dissipative, but they will often converge to spurious modes often betrayed by a slow convergence when grid resolution is increased.

3. Centrifugal effects

The centrifugal force is proportional to Ω^2 whereas the Coriolis force is linear with Ω. One may therefore expect that centrifugal effects are of second order and less important than that of the Coriolis force. This is certainly true for slowly rotating stars like the Sun but likely incorrect for rapidly rotating stars as we shall see now.

The first effect of the centrifugal force is to modify the equilibrium shape of the star by inducing a polar flattening $\mathcal{O}(\Omega^2 R^3/GM)$. So very basically, the fluid's container is changed. This change, although of second order, is very important as it affects mostly the outer layers which contain most of the amplitude of observable modes.

If ε is the polar flattening of the star and R its radius, all modes whose wavelength is of order εR will be sensitive to the ellipsoidal shape of the container; the period of such modes is $\varepsilon R/c_s$

which is ∼25 mn for a star with M=2 $M_\odot$ and R=2 $R_\odot$ rotating at 150 km/s when c_s ∼100 km/s. It is thus clear that modes of δ Scuti stars cannot be calculated without taking fully into account their ellipsoidal shape.

This geometrical constraint of the shape is not the only centrifugal effect; the other effect comes from the impossibility of having radiative zones at rest when the fluid is rotating; this is a consequence of the famous Von Zeipel theorem which implies that fluid motions, like circulation and differential rotation, occur in radiative zones of rotating stars. These motions imply mixing and therefore will be effective during the whole evolution of the stars; as a consequence they will affect the distribution of elements, the existence of μ-barriers, the distribution of angular momentum... All these effects can only be fully appreciated by computing the evolution of the rotating star which is not a simple matter, mainly because the above mentioned motions generate turbulence in a rotating stratified fluid and therefore require some turbulence modelling which is still an uncertain matter.

4. Conclusions

We have shown that the low-frequency part of the spectrum of oscillations, namely gravito-inertial modes, is strongly influenced by the Coriolis force. The role of this force is now well understood and described in detail in Dintrans et al. (1999) or Dintrans & Rieutord (2000). To summarize shortly we may say that the Coriolis force introduces singularities in the low-frequency eigenmodes, singularities which show up as shear layers whose width is controlled by dissipative mechanisms (heat diffusion or viscosity). Because of these layers (and also because of boundary layers), low-frequency eigenmodes are more stable than their nonrotating equivalent and thus are more difficult to excite. On the other hand, internal shear layers may play an interesting part in mixing processes acting in radiative zones.

The centrifugal force has an influence which is presently not as clear. Its first role in changing the shape of the star is currently under study (Lignières and Rieutord 2001) and is suspected to have a great influence on acoustic modes in rapidly rotating stars like δ Scuti stars. Its second, but more subtle, role is its long-term influence during the evolution by the mixing processes it induces. These effects need to be appreciated through simulations of the evolution of a rotating star using 2D models accounting for the transport mechanisms.

Finally, let us mention that as acoustic modes of the Sun or gravity modes of white dwarfs, all the observable modes of a rotating star can be used to constrain its structure; but, as in previous cases, they should be unambigously identified and this requires a zero order model of good quality. The construction of such a model will be all the more difficult that evolution effects are important.

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