Wavelet bases: What they are and what they can do for you

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joint work with

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Turbulent cascade in the solar wind

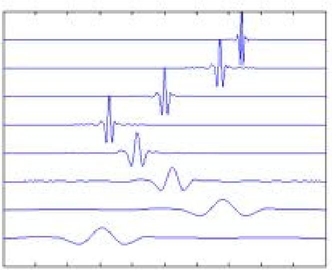
Paris and Meudon Observatories, September 17-21, 2012

One-variable wavelet basis

A wavelet basis on the real line $\mathbb{R}$ is generated by one smooth, well localized, oscillating function ψ such that the

$$\psi(2^j x - k), \quad j, k \in \mathbb{Z}$$

form an orthogonal basis

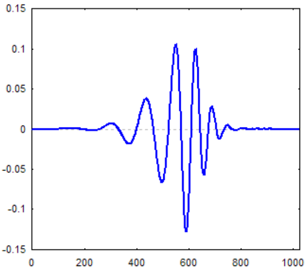


$\forall f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k} \, \psi(2^j x - k)$$

the decomposition is unique

Daubechies Wavelet



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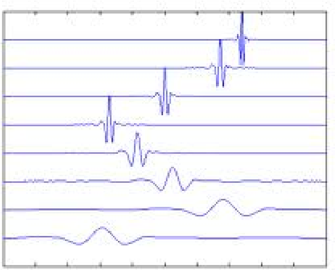

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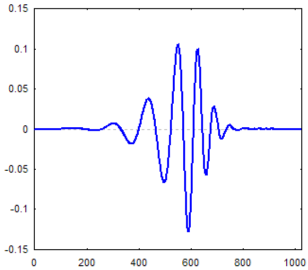
Space-scale decomposition



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Orthogonal basis

Let $f : \mathbb{R} \to \mathbb{R}$. The "energy" of f is $\int |f(x)|^2 \, dx$

Wavelets form an orthogonal basis i.e.

$$\int \psi(2^j x - k)\, \psi(2^{j'} x - k') dx = \begin{cases} 2^{-j} & \text{if } j = j', k = k' \\ 0 & \text{else} \end{cases}$$

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The energy of f can be recovered from the coefficients

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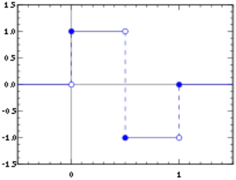
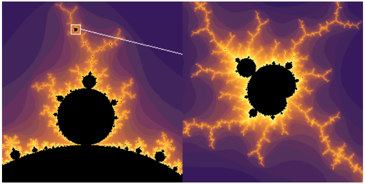
A. Haar, J. O. Strömberg, Y. Meyer, P.-G. Lemarié, S. Mallat, I. Daubechies, ...

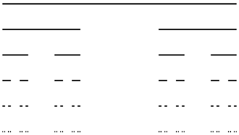


Haar Wavelet (1909)

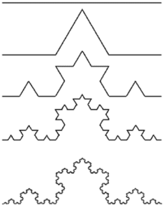
Selfsimilarity : Sets



Mandelbrot Set

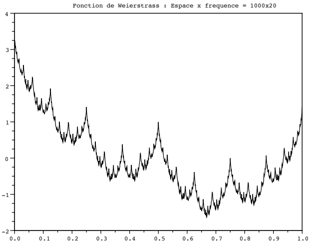


Triadic Cantor set



Van Koch curve

Selfsimilarity : Deterministic functions



Weierstrass-Mandelbrot function

$$W_\alpha(x) = \sum_{j=-\infty}^{+\infty} 2^{-\alpha j} \sin(2^j x)$$

$0 < \alpha < 1$

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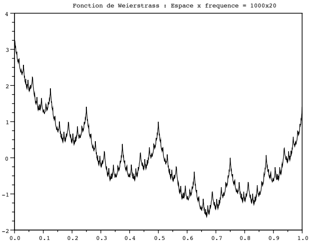

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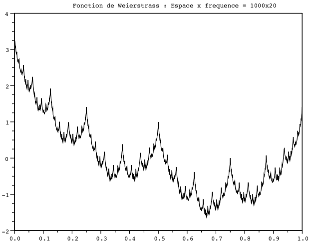

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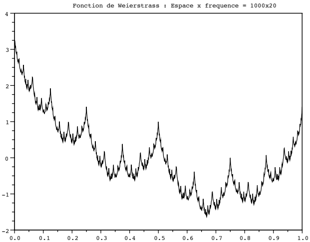

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$$= 2^\alpha W_\alpha(x)$$

Selfsimilarity : Implication for wavelet coefficients

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Selfsimilarity : Random processes

Fractional Brownian motion $B_H(x)$

Centered Gaussian process of covariance :

$$\mathbb{E}(B_H(x)B_H(y)) = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)$$

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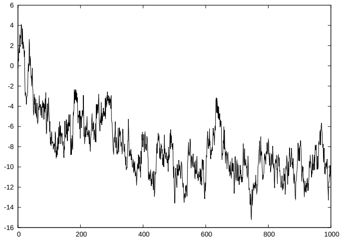

FBM of Hurst exponent

$H = 1/3$

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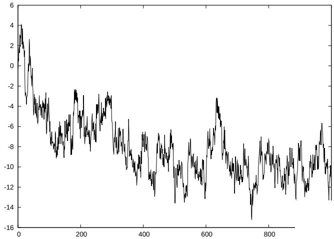

FBM of Hurst exponent

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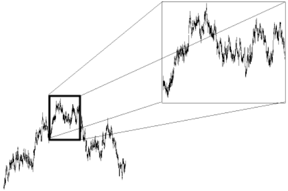
Selfsimilarity in law :

$$B_H(ax) \stackrel{\mathcal{L}}{=} a^H B_H(x)$$



Wavelet coefficients of random processes

Let $X(x)$ be a selfsimilar random process of exponent H
The sequence of random vectors $\mathcal{C}_j \equiv (2^{Hj} c_{j,k})_{k\in\mathbb{Z}}$ satisfies :

The $\{\mathcal{C}_j\}_{j\in\mathbb{Z}}$ share the same law

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FBM of exponent H : The $2^{Hj} c_{j,k}$ (for all j and k) share the same law

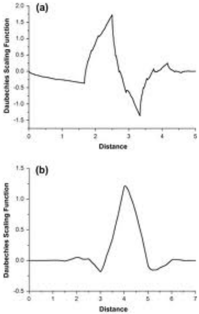
Wavelet bases in two variables

In 2D, the wavelets used are tensor products :

$$\psi^1(x, y) = \psi(x)\varphi(y)$$

$$\psi^2(x, y) = \varphi(x)\psi(y)$$

$$\psi^3(x, y) = \psi(x)\psi(y)$$



Daubechies functions φ

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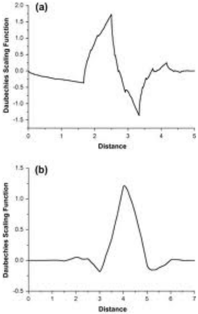

Daubechies functions φ

Vanishing moments

$$\int\int \psi(x,y) x^a y^b \, dx \, dy = 0 \qquad \text{if } a+b \leq N$$

Wavelet bases in two variables

A wavelet basis on $\mathbb{R}^2$ is of the form

$$2^j \psi^i(2^j(x,y)-(k,l)), \qquad i=1,2,3,\ j\in\mathbb{Z},\ (k,l)\in\mathbb{Z}^2$$

Wavelets form an orthogonal basis

$$f(x,y)=\sum_{i,j,k,l} c^i_{j,k,l}\,\psi^i(2^j(x,y)-(k,l))$$

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Wavelet coefficients : $c^i_{j,k,l} = 2^{2j} \int \int f(x, y)\psi^i(2^j(x, y) - (k, l))dx$

Wavelet bases in two variables

Notations

Dyadic squares : $$\lambda = \left[\frac{k}{2^j}, \frac{(k+1)}{2^j}\right] \times \left[\frac{l}{2^j}, \frac{(l+1)}{2^j}\right]$$

Wavelet coefficients

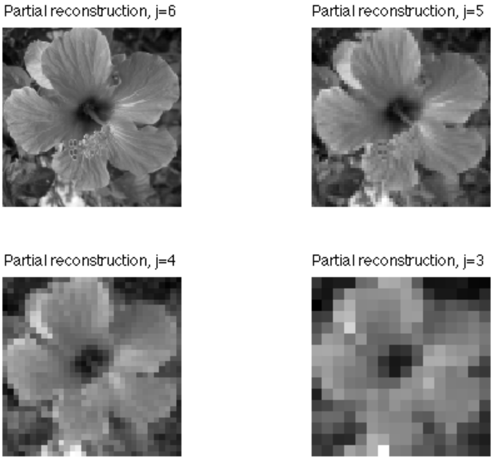
$$c_\lambda = 2^{2j} \int \int f(x, y)\, \psi^i \left(2^j x - k, 2^j y - l\right)\, dx\, dy$$

Computation of 2D wavelet coefficients

Notation for dyadic squares at scale j : $\Lambda_j = \{\lambda : \quad |\lambda| = 2^{-j}\}$

Partial reconstructions of an image :

Two-dimensional Haar wavelet



Credit to : http ://www.ceremade.dauphine.fr/ peyre/numerical-tour/tours/

Notations for wavelets on $\mathbb{R}$

Dyadic intervals

$$\lambda = \left[\frac{k}{2^j}, \frac{k+1}{2^j} \right]$$

Wavelets

$$\psi_\lambda(x) = \psi(2^j x - k)$$

Wavelet coefficients

$$c_\lambda = 2^j \int f(x)\, \psi(2^j x - k)\, dx$$

Dyadic intervals at scale j

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Wavelet expansion of f

$$f(x) = \sum_j \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x)$$

Why use wavelet bases ?

- Characterization of selfsimilarity
- Images are stored by their wavelet coefficients (JPEG 2000)
- Fast decomposition algorithms
- Sparse representations
- Characterization of regularity
- Scaling invariance

Wavelet characterization of uniform regularity

C^α **Spaces** : Let $\alpha \in (0,1)$; $f \in C^\alpha(\mathbb{R})$ if

$$\exists C, \forall x, y : \qquad |f(x) - f(y)| \leq C\, |x - y|^\alpha$$

Wavelet characterization of uniform regularity

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Theorem : A function f belongs to $C^\alpha(\mathbb{R})$ if and only if its wavelet coefficients satisfy

$$\exists C > 0 \;\; \forall j, k, \qquad |c_{j,k}| \leq C\, 2^{-\alpha j}$$

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Local version of the theorem imply that, if f is a piecewise smooth function, then its wavelet expansion is sparse

Fractional Brownian Motions

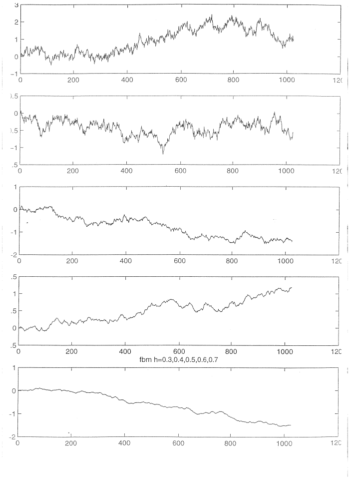
$H = 0.3$

$H = 0.4$

$H = 0.5$

$H = 0.6$

$H = 0.7$



Gaussian

processes

with

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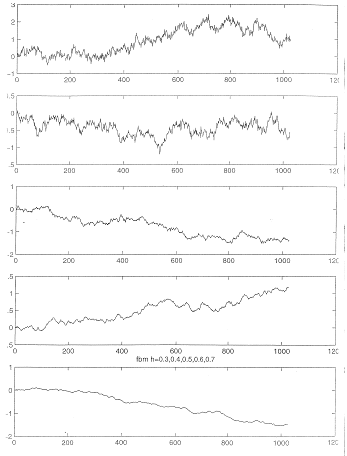

Gaussian processes with stationary increments

Challenge : Find a numerically stable way to decide if a real-life signal can be modeled by FBM

Kolmogorov scaling function (1941)

$$\int |f(x+\delta) - f(x)|^p dx \sim |\delta|^{\zeta_f(p)}$$

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Regression on a log-log plot

What is the scaling function of FBM ?

B_H is the unique centered Gaussian process such that

$$\forall x, \delta \geq 0, \qquad \mathbb{E}(|B_H(x+\delta) - B_H(x)|^2) = |\delta|^{2H}$$

It follows that

$$|B_H(x+\delta) - B_H(x)| \sim |\delta|^H$$

$$\int |B_H(x+\delta) - B_H(x)|^p dx \sim |\delta|^{Hp}$$

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$\Longrightarrow$ Turbulence at small scale cannot be modeled by FBM (1950s)

The wavelet scaling function

The wavelet scaling function is defined by

$$\forall p > 0 \qquad 2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

$$\zeta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \right)}{\log(2^{-j})}$$

Theorem : when $p \geq 1$, the wavelet scaling function coincides with Kolmogorov's scaling function

The scaling function yields a smoothness index in the L^p norm

The roles of the wavelet scaling function :

I : Function space regularity

- If $\zeta_f(1) > 1$, then $f \in BV$
- If $\zeta_f(1) < 1$, then $f \notin BV$
- If $\zeta_f(1) > 0$, then f is a measure
- If $\zeta_f(1) < 0$, then f is not a measure
- If $\zeta_f(2) > 0$, then $f \in L^2$
- If $\zeta_f(2) < 0$, then $f \notin L^2$

Motivation :

Y. Gousseau, J.-M. Morel : *Are natural images of bounded variation ?* (SIAM J. Math. Anal, Vol. 3 2001)

The wavelet scaling function is independent of the (smooth enough) wavelet basis chosen

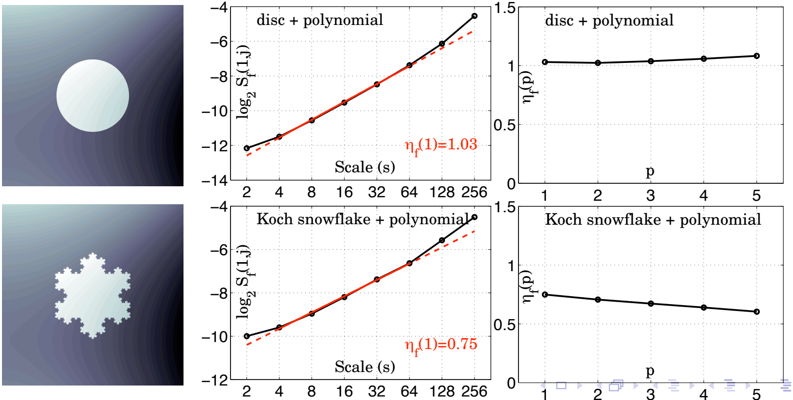
It is defined by regressions on log-log plots

Wavelet scaling functions of synthetic images

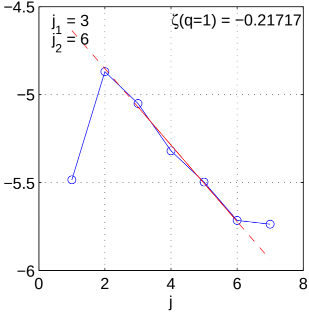
Wavelet scaling function $\zeta_f(p)$:

$$2^{-2j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p)\, j}$$

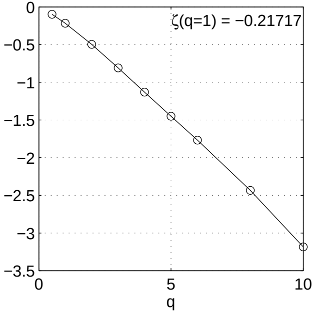
Disk : $\zeta_f(p) = 1$ Van Koch snowflake : $\zeta_f(p) = 2 - \dfrac{\log 4}{\log 3} \sim 0.74$

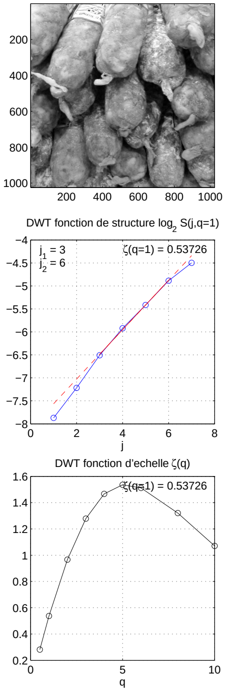


Natural images



DWT fonction d'echelle ζ(q)




DWT fonction de structure log2 S(j,q=1)
j1 = 3
j2 = 6
ζ(q=1) = 0.53726
j
DWT fonction d'echelle ζ(q)
ζ(q=1) = 0.53726
q

Uniform Hölder regularity

Hölder spaces : Let $\alpha \in (0,1)$; $f \in C^\alpha$ if

$$\exists C, \ \ \forall x, y : \qquad |f(x) - f(y)| \leq C \cdot |x-y|^\alpha$$

The uniform Hölder exponent of f is

$$H_f^{min} = \sup\{\alpha : \ f \in C^\alpha\}$$

$$f \in C^\alpha \iff \exists C > 0, \ \ \forall \lambda \qquad |c_\lambda| \leq C \, 2^{-\alpha j}$$

Numerical computation

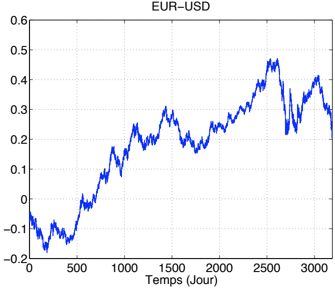
Let $\omega_j = \sup_{\lambda \in \Lambda_j} |c_\lambda|$ then $H_f^{min} = \liminf_{j \to +\infty} \dfrac{\log(\omega_j)}{\log(2^{-j})}$

$$H_f^{min} > 0 \implies f \text{ is continuous}$$

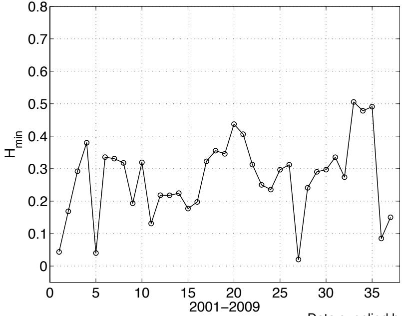
$$H_f^{min} < 0 \implies f \text{ is not locally bounded}$$

Validity of jump models in finance

Euro vs. USD
2001-2009



H_f^{min}

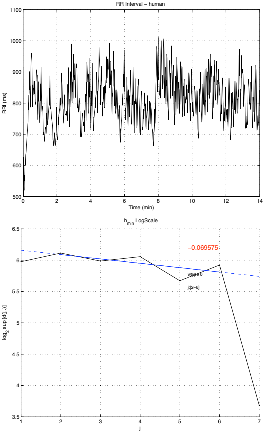


Data supplied by Vivienne Investissement

Classification based on the uniform Hölder exponent

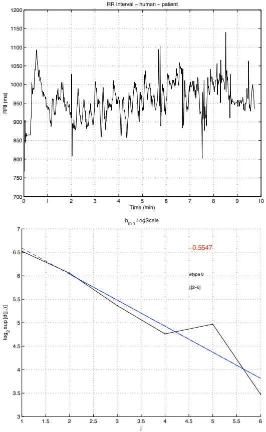
Heartbeat intervals

Healthy



$H_f^{min} = -0.06$

Heartbeat failure



$H_f^{min} = -0.55$

Limitations

Classification only based on the wavelet scaling function or on the uniform Hölder exponent proved insufficient in several occurrences (turbulence, data mining, ...). This motivated further developments based on seminal ideas introduced by U. Frisch and G. Parisi, which paved the way to the construction of new scaling functions

Pointwise regularity

Let f be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $x_0 \in \mathbb{R}^d$; $f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : \quad f \in C^\alpha(x_0)\}$$

Heuristics : $\quad |f(x) - f(x_0)| \sim |x - x_0|^{h_f(x_0)}$

Constant Hölder exponents

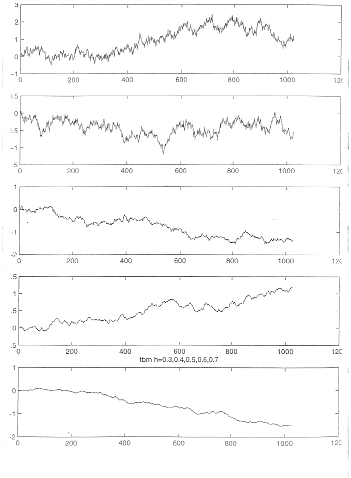
$H = 0.3$

$H = 0.4$

$H = 0.5$

$H = 0.6$

$H = 0.7$



Gaussian

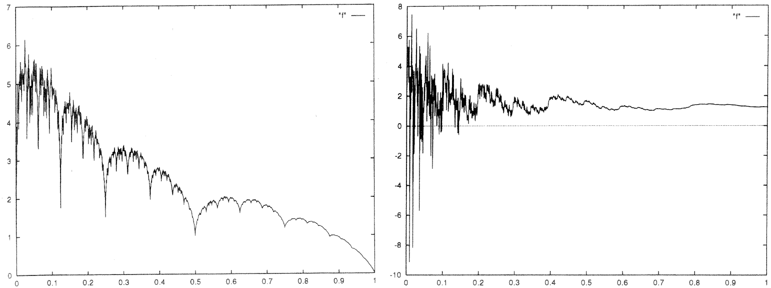
processes

with

stationary

increments

Varying Hölder exponent

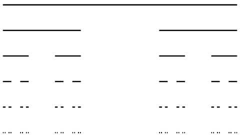


$h(x) = x$
(Constructions of K. Daoudy, J. Lévy-Véhel and Y. Meyer)

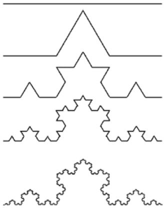
Fractional dimension

The fractional dimension of a set $A \subset \mathbb{R}^d$ is defined by the following property :
Let $N(\varepsilon)$ denote the minimal number of balls of radius ε required to cover the set A. Then

$$N(\varepsilon) \sim \varepsilon^{-dim(A)}$$



Triadic Cantor set



Van Koch curve

Dimensions : $\dfrac{\log 2}{\log 3}$ $\dfrac{\log 4}{\log 3}$

Multifractal spectrum (Parisi and Frisch, 1985)

The isohölder sets of f are the sets

$$E_H = \{x_0 : \quad h_f(x_0) = H\}$$

Let f be a locally bounded function. The multifractal spectrum of f is

$$D_f(H) = \dim\,(E_H)$$

where dim stands for the fractional dimension

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Parisi and Frisch's fundamental idea was that the nonlinearity of the scaling function reflects the presence of a whole range of fractal sets E_H, and that the scaling function yields information on the "sizes" of these sets.

Can one estimate the spectrum of singularities of signals and images ?

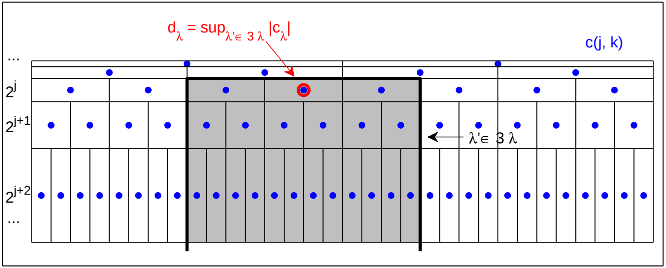
Can it yield new classification parameters ?

Wavelet leaders

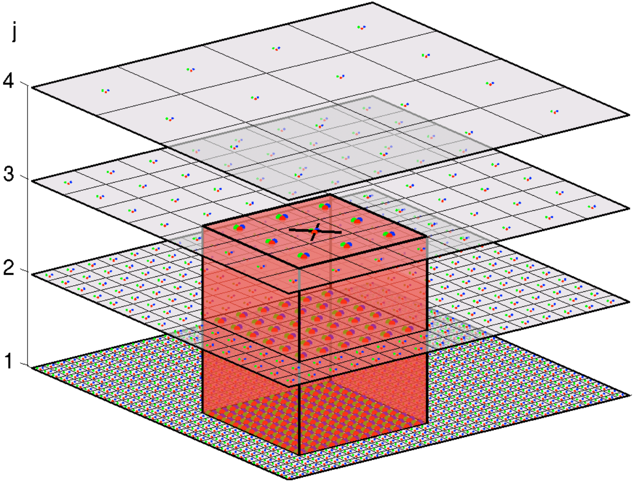
Let λ be a dyadic cube ; 3λ is the cube of same center and three times wider

Let f be a locally bounded function ; the wavelet leaders of f are

$$d_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|$$



Computation of 2D wavelet leaders



Wavelet leaders allow to estimate the pointwise Hölder exponent

Leader scaling function

Λ_j denotes the set of dyadic cubes of width 2^{-j}

Wavelet scaling function

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p \sim 2^{-\zeta_f(p) j}$$

Leader scaling function

$$2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \sim 2^{-\eta_f(p) j}$$

$$\forall p \in \mathbb{R}, \qquad \eta_f(p) = \liminf_{j \to +\infty} \frac{\log \left(2^{-dj} \sum |d_\lambda|^p \right)}{\log(2^{-j})}$$

Advantages :

- If $\psi^i \in \mathcal{S}(\mathbb{R}^d)$, then $\forall p \in \mathbb{R}$, $\eta_f(p)$ is independent of the wavelet basis
- η_f is invariant under "smooth perturbations" of f

Multifractal formalism

Since η_f is a concave function, there is no loss of information in rather considering its Legendre transform :

The Legendre Spectrum of f is

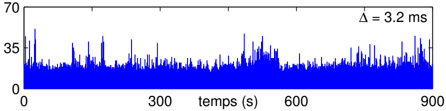
$$L_f(H) = \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

Theorem : Let $D_f(H)$ denote the Hausdorff dimension of the set of points where $h_f(x) = H$. If $f \in C^\varepsilon(\mathbb{R}^d)$ for an $\varepsilon > 0$ then

$$\forall H \in \mathbb{R}, \qquad D_f(H) \leq \inf_{p \in \mathbb{R}} (d + Hp - \eta_f(p))$$

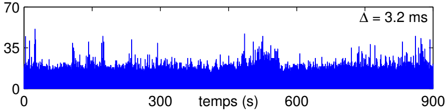
The multifractal formalism is satisfied when equality holds

Monohölder vs. Multifractality

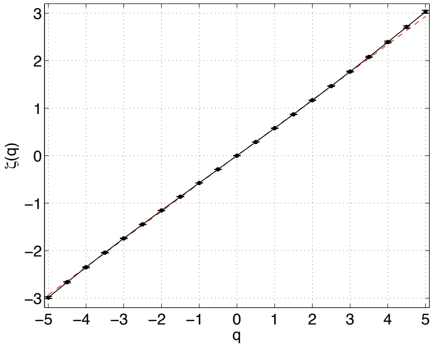


Internet traffic

Monohölder vs. Multifractality



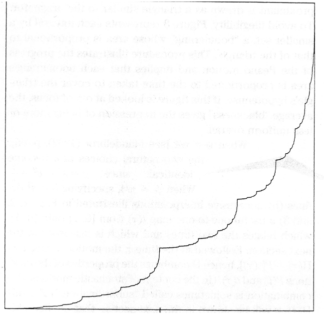
Internet traffic



Wavelet leader scaling function

Data from http ://mawi.wide.ad.jp/mawi/

Cascade models : Binomial cascade

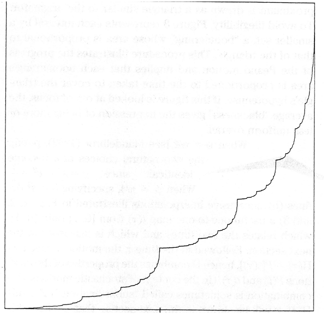


We obtain a measure μ on the interval $[0, 1]$:

$$\mu(I) = \text{ amount of sand that fell on } I$$

Intervals of length $\left(\frac{1}{2}\right)^n$:

Cascade models : Binomial cascade



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Intervals of length $\left(\frac{1}{2}\right)^n$:

far left : $\mu(I) = \left(\frac{1}{4}\right)^n = |I|^2$ far right : $\mu(I) = \left(\frac{3}{4}\right)^n = |I|^{\log(4/3)/\log 2}$

average : $\mu(I) = \left(\frac{1}{4}\right)^{n/2}\left(\frac{3}{4}\right)^{n/2} = |I|^{\log(4/\sqrt{3})/\log 2}$

Cascade models : Binomial cascade

Repartition function of the measure μ :
$f(x) = \mu([0,x]) =$ amount of sand in $[0,x]$

$$f(x+\delta) - f(x) = \mu([x, x+\delta]) \sim \delta^{h(x)}$$

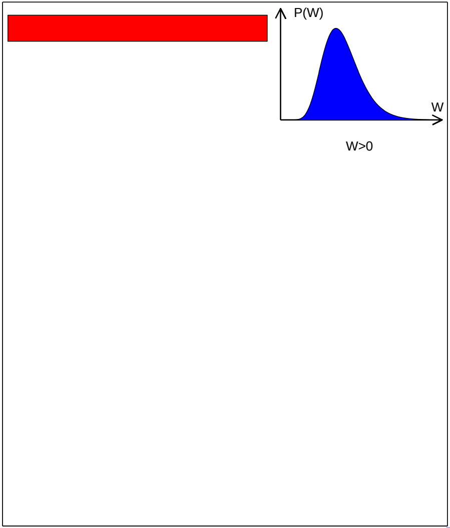
$$h(x) \in \left[\frac{\log(4/3)}{\log 2}, 2\right]$$

Spectrum of singularities of f :

$$d_f(H) = \dim(\{x : h(x) = H\})$$

Model refutation

Multiplicative cascades



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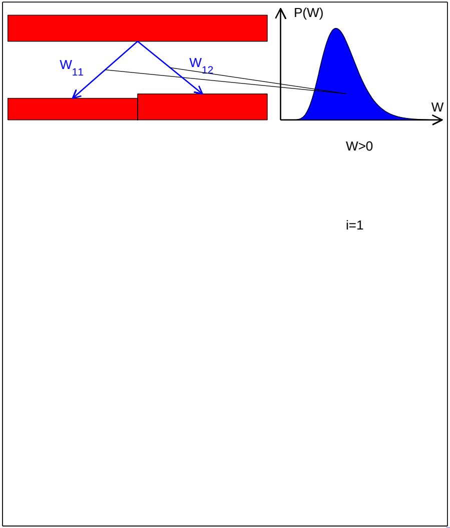

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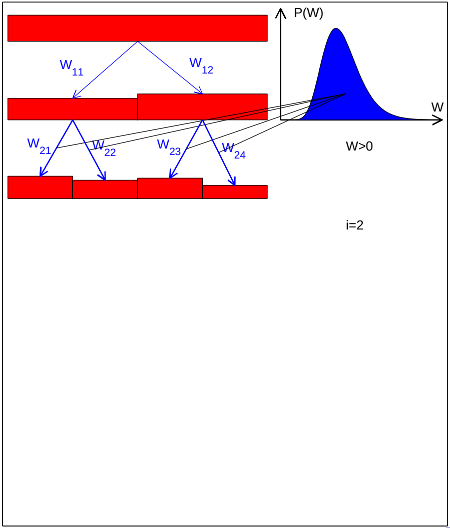

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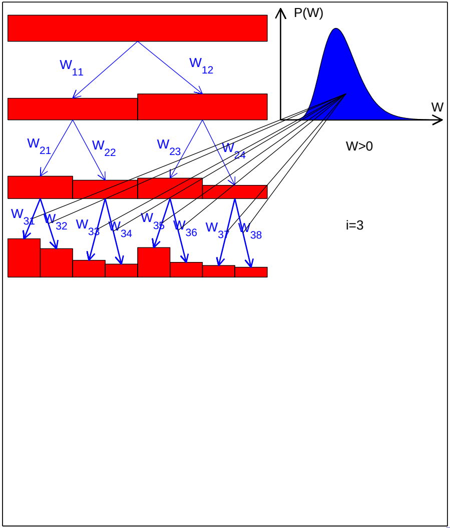

Model refutation

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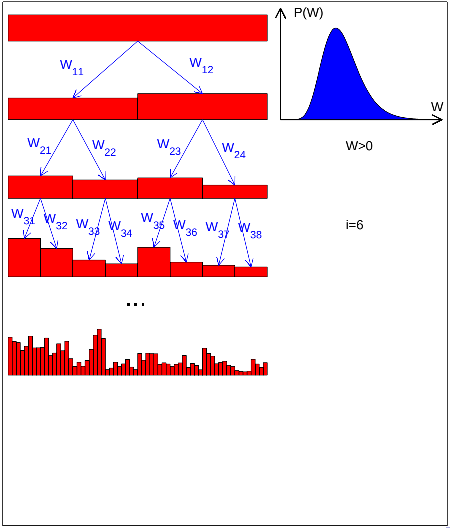

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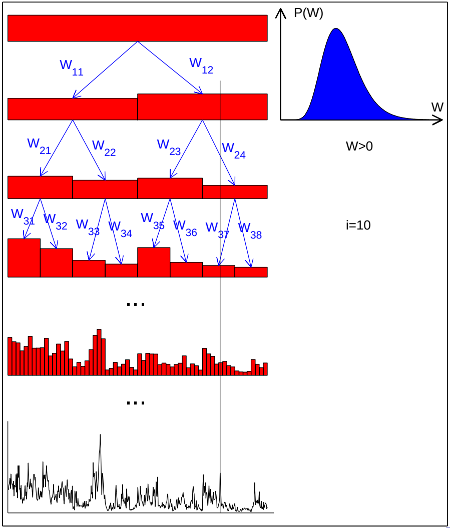

Processes based on multiplicative cascades

A cascade is a positive measure, and therefore not appropriate to model oscillating signals

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Let $F(x)$ be the repartition function associated with the measure μ,

$$F(x) = \int_{-\infty}^{x} d\mu$$

The FBM in multifractal time is

$$X(t) = B_H(F(t))$$

proposed by Calvet, Fisher and Mandelbrot in financial modeling

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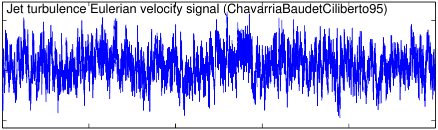
Wavelet cascades defined by their wavelet coefficients

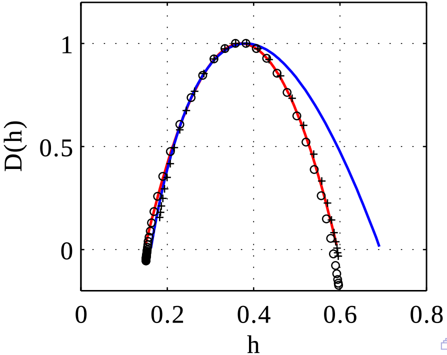
$$c_\lambda = 2^{-\beta j}\mu(\lambda)^q$$

Model refutation : Fully developed turbulence

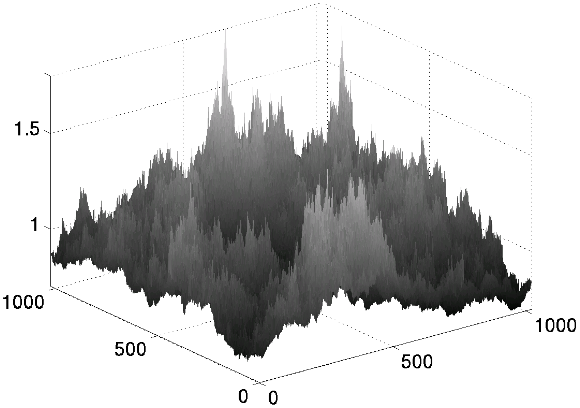
(joint work with Bruno Lashermes)



Log-normal vs. Log-Poisson model

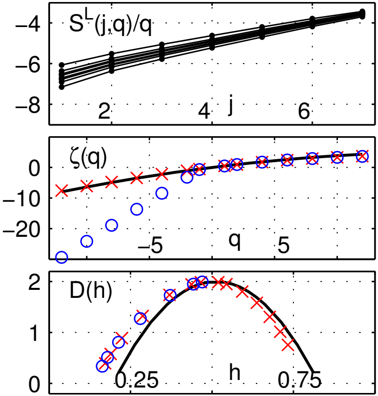


Binomial Log-normal 2-D Cascade

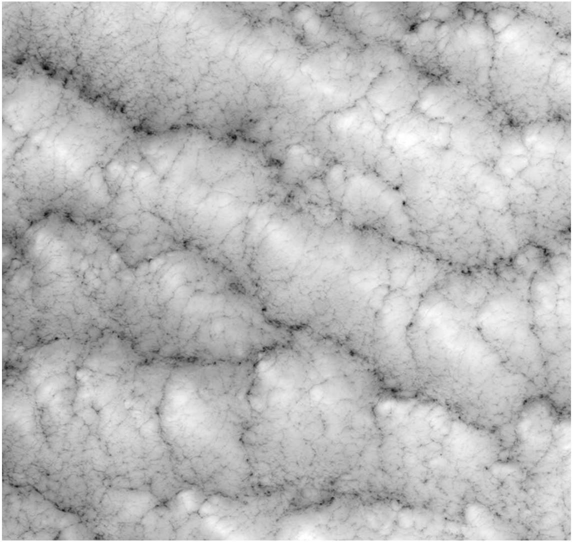


Binomial Log-normal 2-D Cascade

Scaling function and Legendre spectrum



Satellite Image of cloud



Satellite Image of cloud

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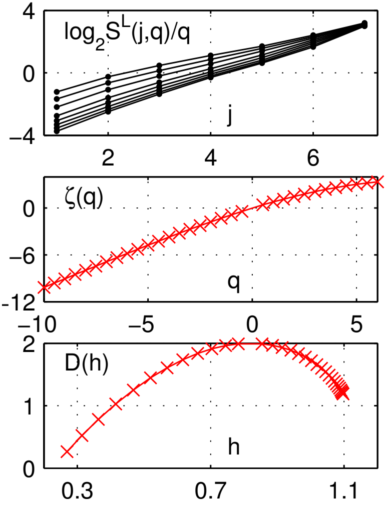
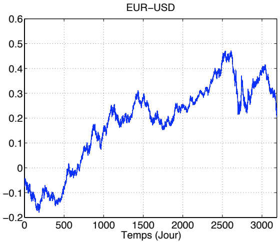


Evidence of time-evolution : Finance

Multifractal analysis of the USD-Euro change rate



Legendre spectra