Which shell model,
for what purpose ?

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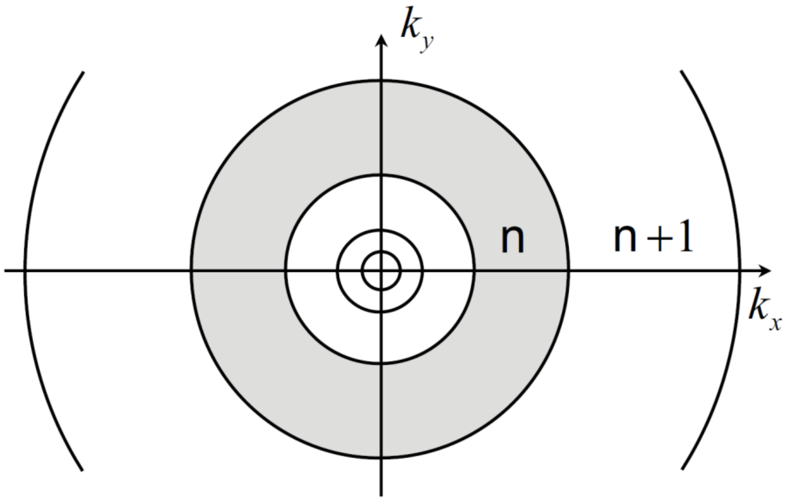
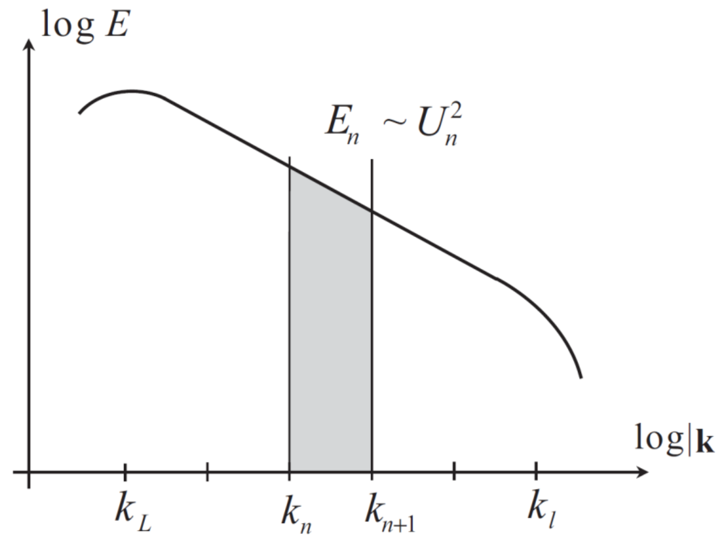
Shell models of MHD turbulence





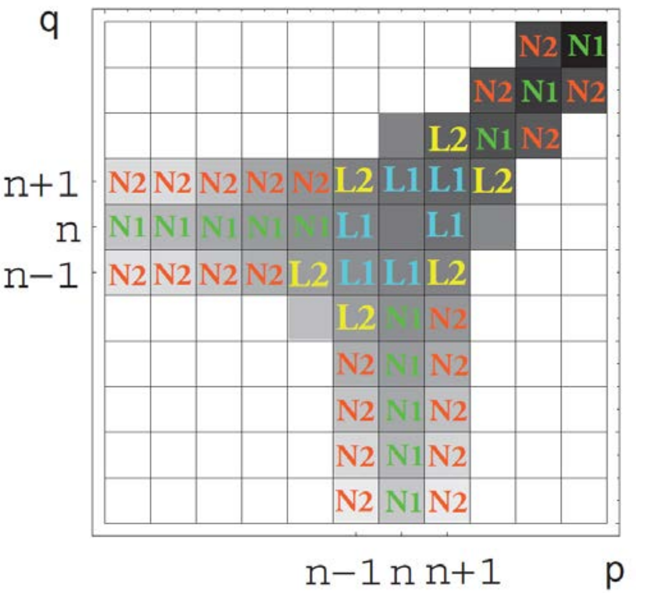
- $$\begin{cases} d_t U_n = \tilde{W}_n(U,U) - \tilde{W}_n(B,B) - \nu k_n^2 U_n + F_n \\ d_t B_n = \tilde{W}_n(U,B) - \tilde{W}_n(B,U) - \eta k_n^2 B_n \end{cases}$$

- $$\tilde{W}_n(X,Y) = k_n \sum_{i,j=1}^{N} a_{ij} X_i Y_j + b_{ij} X_i^* Y_j + c_{ij} X_i Y_j^* + d_{ij} X_i^* Y_j^*$$

- $$\tilde{W}(X,Y) \cdot Y = 0 \quad \Rightarrow d_t \sum_n \left(|U_n|^2 + |B_n|^2 \right) = d_t \sum_n \left(U_n^* B_n + B_n^* U_n \right) = 0$$

MHD GOY model

- $$\begin{cases} d_t U_n = \tilde{W}_n(U,U) - \tilde{W}_n(B,B) - \nu k_n^2 U_n + F_n \\ d_t B_n = \tilde{W}_n(U,B) - \tilde{W}_n(B,U) - \eta k_n^2 B_n \end{cases}$$



- $$\tilde{W}_n(X,Y) = k_n \sum_{i,j=1}^{N} d_{ij} X_i^* Y_j^*$$

- Two - first neighbor interactions (L2)

- $$\tilde{W}(X,Y) \cdot Y = 0 \quad \Rightarrow d_t \sum_n \left(|U_n|^2 + |B_n|^2 \right) = d_t \sum_n \left(U_n^* B_n + B_n^* U_n \right) = 0$$

- Additional quadratic invariant ?

Additional quadratic invariant in 2D

HD	MHD
$\Xi = \sum_n k_n^2 \lvert U_n \rvert^2$	$A = \sum_n k_n^{-2} \lvert B_n \rvert^2$

Application to RMHD turbulence (Nigro et al. 2004)

$$\left(\partial_t \mp b_0 \partial_x\right) Z_n^{\pm} = \tilde{W}_n(Z_n^{\mp}, Z_n^{\pm}) - k_n^2 \left(r^+ Z_n^{\pm} + r^- Z_n^{\mp}\right)$$

$$Z_n^{\pm} = U_n \pm B_n$$

- Coronal heating (Nigro et al. 2004, Buchlin & Velli 2007)
- Weak / Strong turbulence (Verdini & Grappin 2012)
- Sub-Alfvénic solar wind spectrum (Verdini et al. 2012)

Additional quadratic invariant in 3D

HD	MHD
$H^U = \sum_n (-1)^n k_n \lvert U_n \rvert^2$	$H^B = \sum_n (-1)^n k_n^{-1} \lvert B_n \rvert^2$

H^U (and H^B) « presents an asymmetry between odd and even shells that does not have any counterpart in physical flow »
Biferale & Kerr (1995)

Additional quadratic invariant in 3D

Two ways to define appropriate helicities:

1. Take more terms in

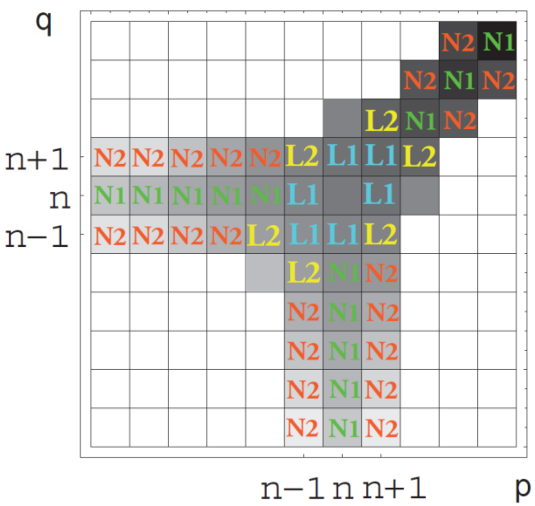
$$\widetilde{W}_n(X,Y) = k_n \sum_{i,j=1}^{N} a_{ij} X_i Y_j + b_{ij} X_i^* Y_j + c_{ij} X_i Y_j^* + d_{ij} X_i^* Y_j^*$$

$\Rightarrow$ Helical L1 model

2. Introduce helical modes $U_n^{\pm}, B_n^{\pm}$

$\Rightarrow$ Helical L2$^{\pm}$ model

Helical L1 model



$$
\begin{aligned}
\widetilde{W}_n(\mathbf{X},\mathbf{Y}) \;=\;& ik_n\Big[(X_{n-1}Y_{n-1}+X^*_{n-1}Y^*_{n-1})-\lambda X^*_nY^*_{n+1}-\frac{\lambda^2}{2}(X_nY_{n+1}+X_{n+1}Y_n)\\
&-\frac{\lambda}{2}(X^*_{n-1}Y_{n-1}-X_{n-1}Y^*_{n-1})+\lambda X^*_nY_{n+1}-\frac{\lambda^2}{2}(X_nY^*_{n+1}+X^*_{n+1}Y_n)\Big]\\
-\;& ick_n\Big[\frac{1}{2}(X_{n-1}Y_n+X_nY_{n-1})+\lambda X^*_nY^*_{n-1}-\lambda^2(X_{n+1}Y_{n+1}+X^*_{n+1}Y^*_{n+1})\\
&+\frac{1}{2}(X_nY^*_{n-1}+X^*_{n-1}Y_n)-\lambda X^*_nY_{n-1}+\frac{\lambda}{2}(X^*_{n+1}Y_{n+1}-X_{n+1}Y^*_{n+1})\Big]
\end{aligned}
$$

Mizeva et al. (2009)

HD	MHD
$H^U = i\sum_n k_n\left(U_n^{*2}-U_n^2\right)$	$H^B = i\sum_n k_n^{-1}\left(B_n^{*2}-B_n^2\right)$

=> Application to magnetic / cross helicity spectra

Helical L1 model

Influence of magnetic helicity injection

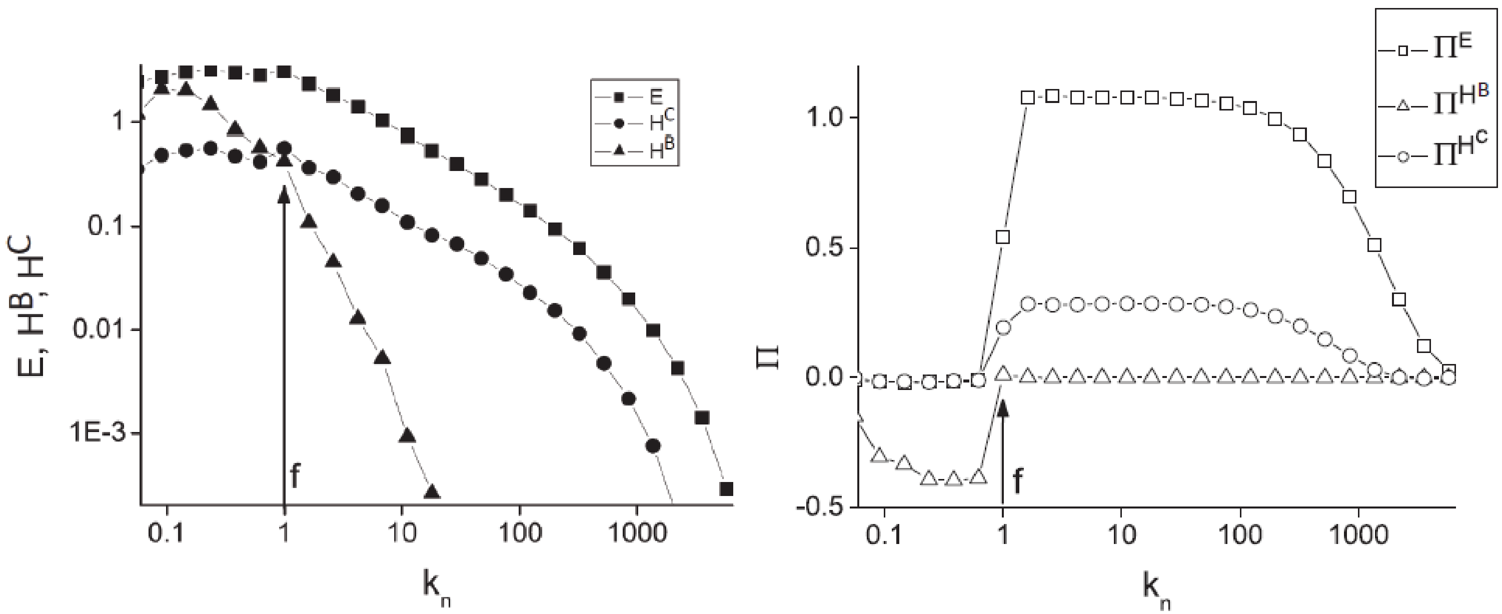


Figure 18: Energy, cross helicity, magnetic helicity spectra (left panel) and fluxes (right panel) of forced MHD turbulence with constant injected helicities ($\epsilon = 1, \chi = 0.3, \xi = 0.4$).

Inverse cascade of magnetic helicity

Helical L1 model

Influence of cross-helicity injection

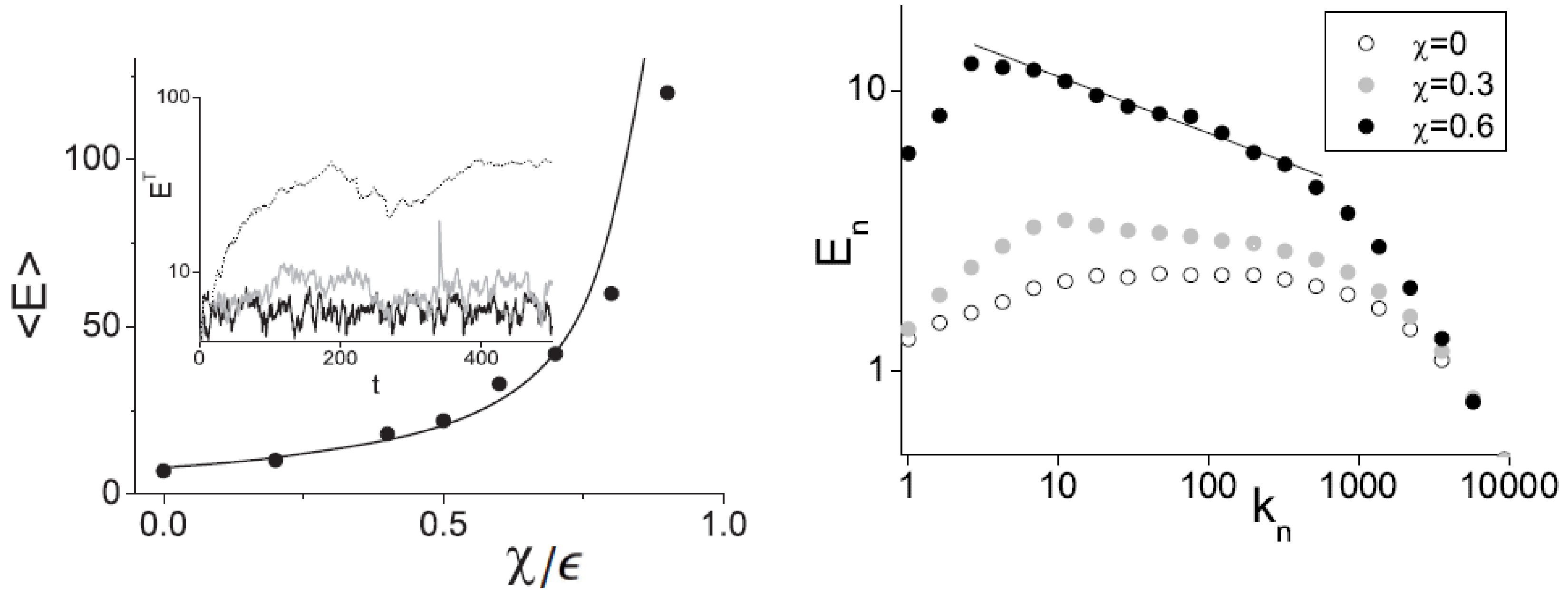
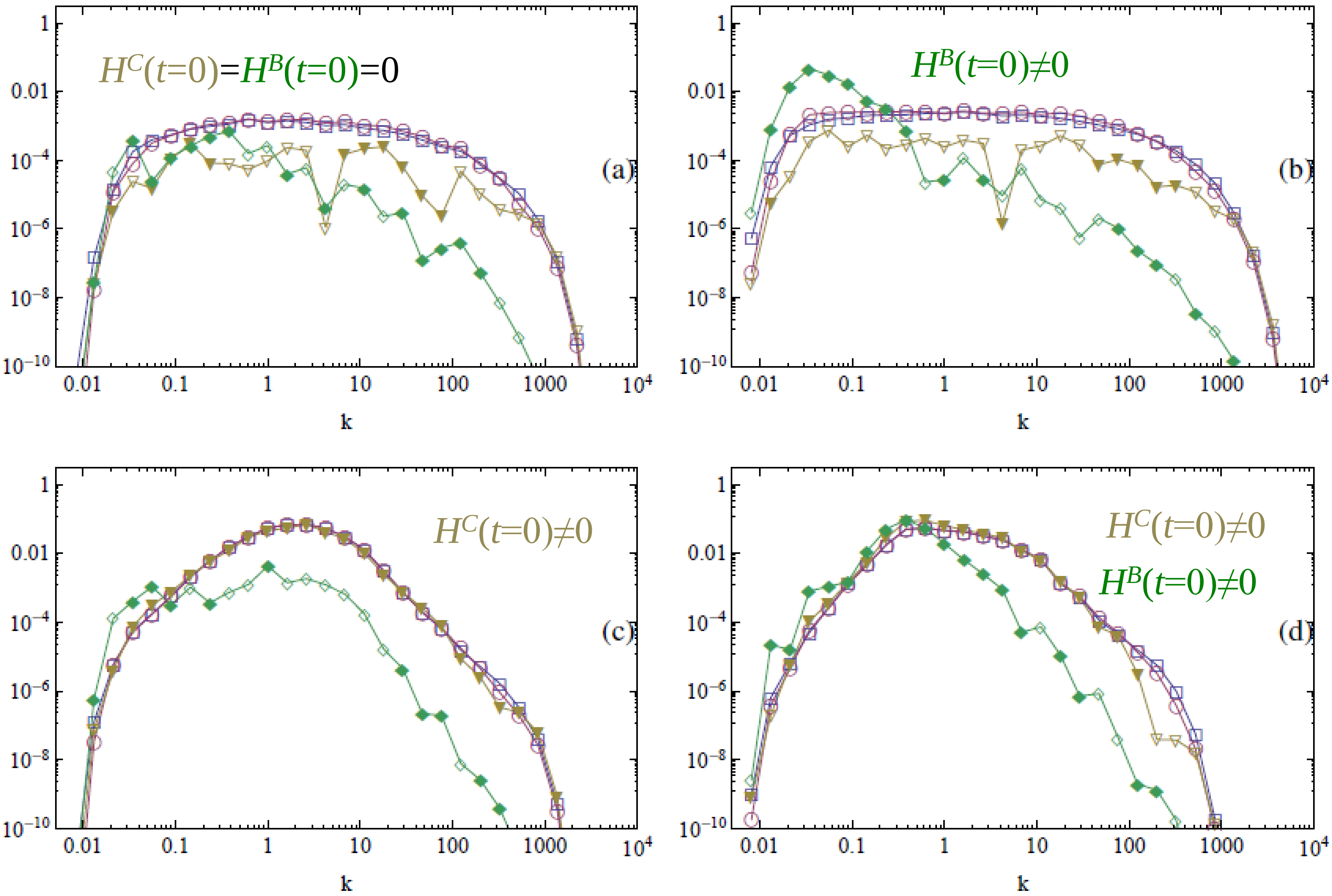


Figure 17: Left panel: Total energy of stationary forced MHD turbulence versus the relative cross helicity injection rate χ/ϵ. The time evolution of energy is shown in the inset, for $\chi = 0$ (thick line), $\chi = 0.3$ (grey line), and $\chi = 0.6$(thin line). Right panel: Total energy spectrum normalized by $k^{-2/3}$ for three different injection rates of cross helicity. The straight line corresponds to $E(k) \propto k^{-1.9}$. Adapted from Mizeva et al. (2009).

$\Rightarrow$ similar to geometrical alignement in real MHD

Helical L1 model

Free-decay for several cross / magnetic helicity initial conditions



Helical L1 model

Free-decay
128 realizations with $H^C/E(t=0) < 10^{-4}$

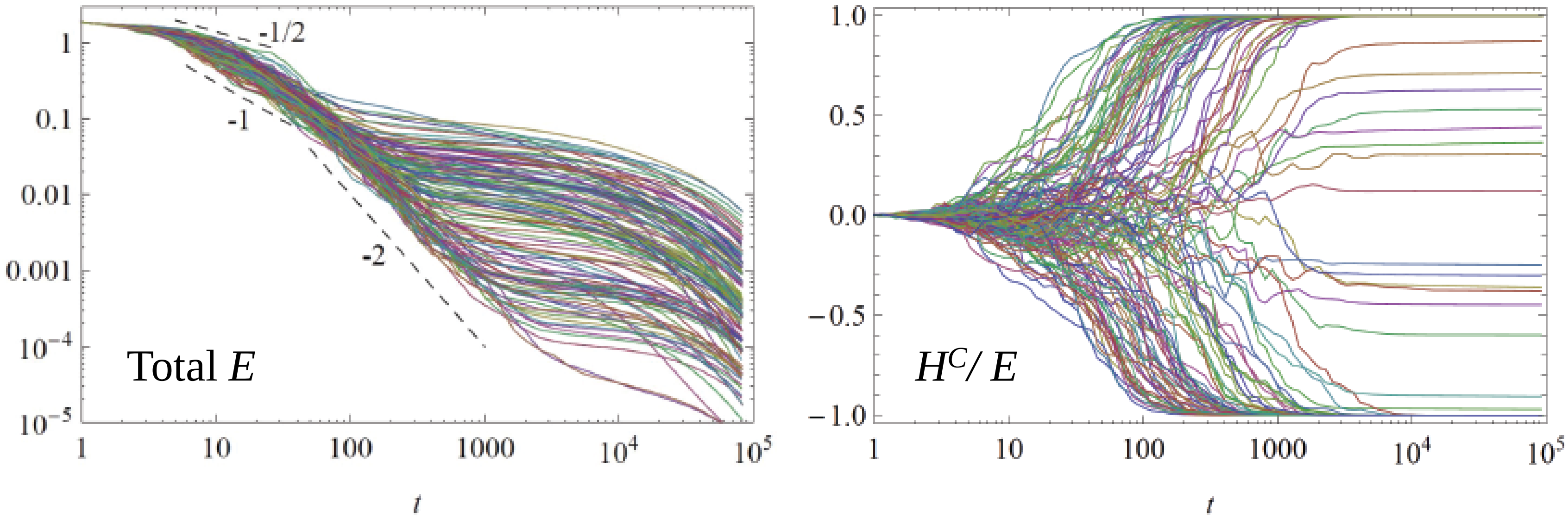


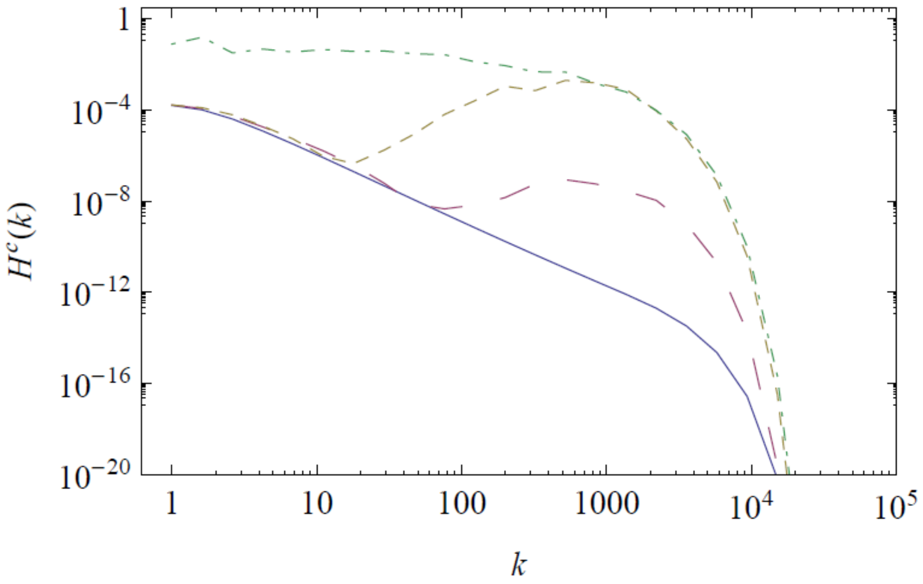
Figure 24: Free decay of total energy E (left panel) and normalized cross helicity H^C/E (right panel) for Pm = 1. The 128 realizations differ by their initial conditions. From Frick and Stepanov (2010).

Where does cross-helicity come from ?

Helical L1 model

Free-decay

128 realizations with $H^C/E(t=0) < 10^{-4}$



From dissipation !

Additional quadratic invariant in 3D

Two ways to define appropriate helicities:

1. Take more terms in

$$\widetilde{W}_n(X,Y) = k_n \sum_{i,j=1}^{N} a_{ij} X_i Y_j + b_{ij} X_i^* Y_j + c_{ij} X_i Y_j^* + d_{ij} X_i^* Y_j^*$$

$\Rightarrow$ Helical L1 model

2. Introduce helical modes $U_n^{\pm}, B_n^{\pm}$

$\Rightarrow$ Helical L2$^{\pm}$ model

Helical L2$^{\pm}$ model

- $$\begin{cases} d_t U_n^{\pm} = \widetilde{W}_n^{\pm}(U,U) - \widetilde{W}_n^{\pm}(B,B) - \nu k_n^2 U_n^{\pm} + F_n^{\pm} \\ d_t B_n^{\pm} = \widetilde{W}_n^{\pm}(U,B) - \widetilde{W}_n^{\pm}(B,U) - \eta k_n^2 B_n^{\pm} \end{cases}$$

- $$\widetilde{W}_n^{\pm}(X,Y) = k_n \sum_{\substack{i,j=1 \\ s_1, s_2 = \pm 1}}^{N} a_{ij}^{s_1 s_2} \left(X_i^{s_1} Y_j^{s_2} \right)^*$$

Biferale & Kerr (1995)
Benzi et al. (1996)

	HD	MHD
2D	$\Xi = \sum_n k_n^2 \left(\left\| U_n^+ \right\|^2 + \left\| U_n^- \right\|^2 \right)$	$A = \sum_n k_n^{-2} \left(\left\| B_n^+ \right\|^2 + \left\| B_n^- \right\|^2 \right)$
3D	$H^U = \sum_n k_n \left(\left\| U_n^+ \right\|^2 - \left\| U_n^- \right\|^2 \right)$	$H^B = \sum_n k_n^{-1} \left(\left\| B_n^+ \right\|^2 - \left\| B_n^- \right\|^2 \right)$

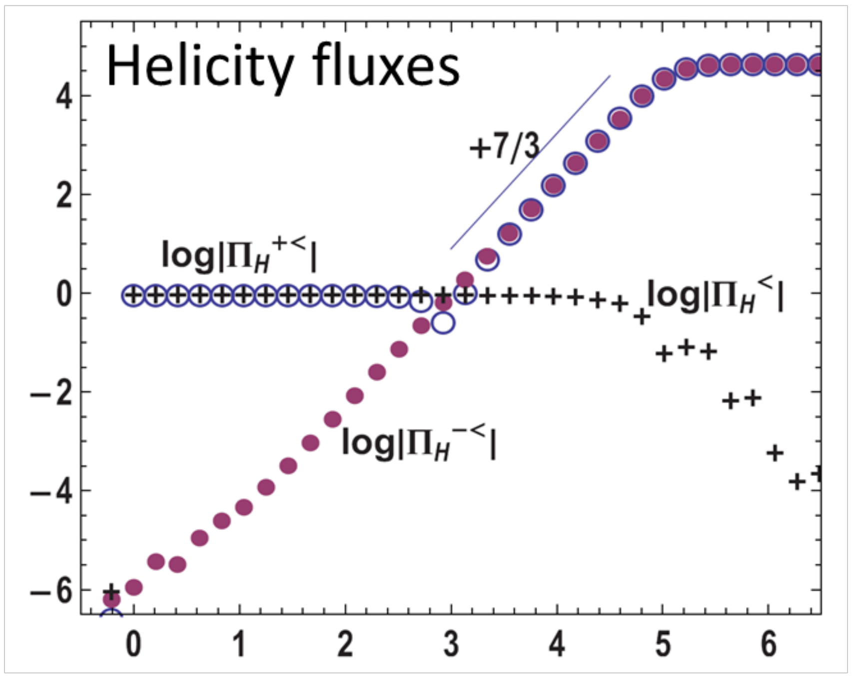
=> Application to helicity spectral flux in HD

Helical L2$^{\pm}$ model

$$\widetilde{W}_n^{\pm}(\mathbf{X},\mathbf{Y}) = \frac{\mathrm{i}}{2}\Big\{k_{n+1}(1+\lambda)(X_{n+1}^{\pm}Y_{n+2}^{\mp} + Y_{n+1}^{\pm}X_{n+2}^{\mp}) - \lambda^{-1}k_{n+1}(X_{n+1}^{\pm}Y_{n+2}^{\mp} - Y_{n+1}^{\pm}X_{n+2}^{\mp})$$
$$-k_n(\lambda^{-1}+\lambda)(X_{n-1}^{\pm}Y_{n+1}^{\mp} + Y_{n-1}^{\pm}X_{n+1}^{\mp}) + k_n(X_{n-1}^{\pm}Y_{n+1}^{\mp} - Y_{n-1}^{\pm}X_{n+1}^{\mp})$$
$$+k_{n-1}(\lambda^{-1}-1)(X_{n-2}^{\mp}Y_{n-1}^{\mp} + Y_{n-2}^{\mp}X_{n-1}^{\mp}) + \lambda k_{n-1}(X_{n-2}^{\mp}Y_{n-1}^{\mp} - Y_{n-2}^{\mp}X_{n-1}^{\mp})\Big\}^*$$

	HD	MHD
2D	$\Xi = \sum_n k_n^2\left(\lvert U_n^+\rvert^2 + \lvert U_n^-\rvert^2\right)$	$A = \sum_n k_n^{-2}\left(\lvert B_n^+\rvert^2 + \lvert B_n^-\rvert^2\right)$
3D	$H^U = \sum_n k_n\left(\lvert U_n^+\rvert^2 - \lvert U_n^-\rvert^2\right)$	$H^B = \sum_n k_n^{-1}\left(\lvert B_n^+\rvert^2 - \lvert B_n^-\rvert^2\right)$

Helical L2$^{\pm}$ model



Lessines etal (2011)

- For each helical mode

$$\Pi_H^{\pm}(k) = \delta^{\pm} \mp C\nu\varepsilon^{2/3}k^{7/3}$$

Dissipation effect => $k_{H^{\pm}} < k_{\nu}$

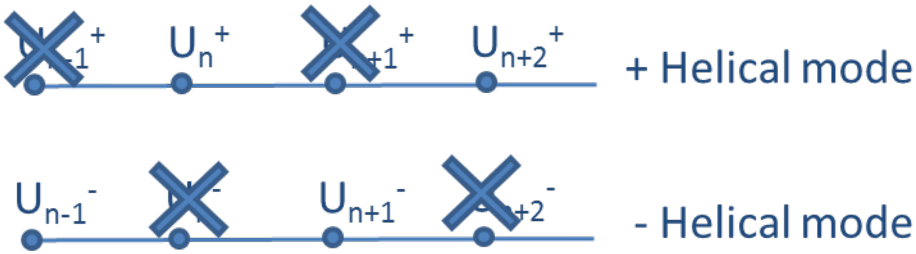
- Total helicity flux

$$\Pi_H(k) = \Pi_H^{+}(k) + \Pi_H^{-}(k)$$

Dissipation effects cancel
=> $k_H = k_{\nu}$

⇒ Turbulence stays helical all along the inertial range

HD GOY model



- For each helical mode

$$\Pi_H^\pm(k) = \delta^\pm \mp C\nu\varepsilon^{2/3}k^{7/3}$$

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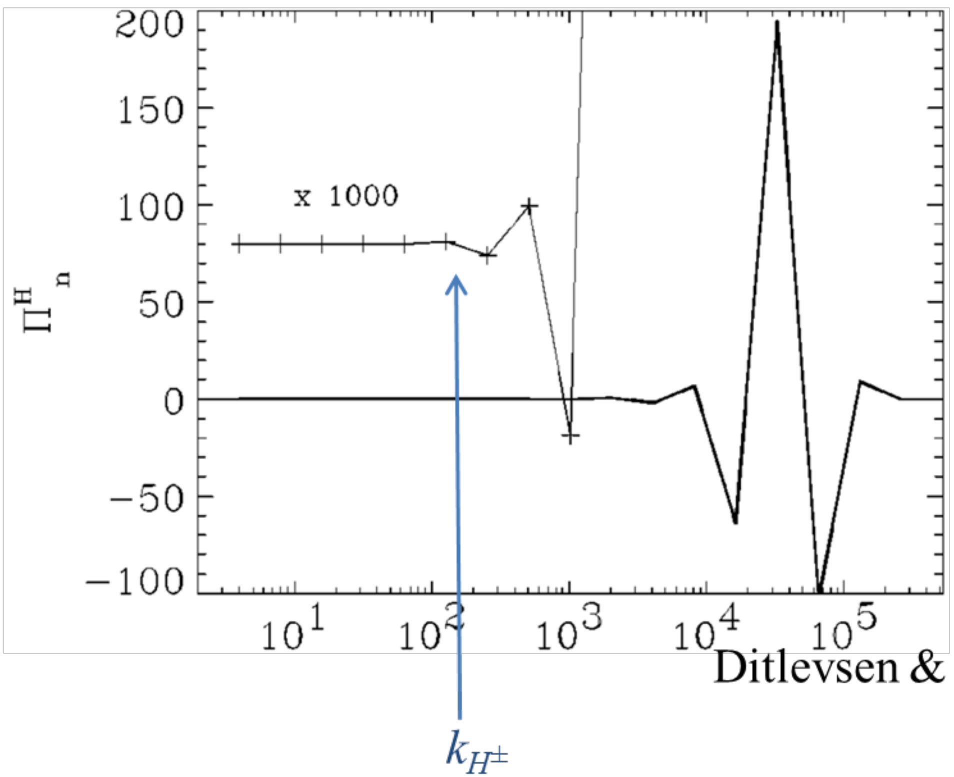
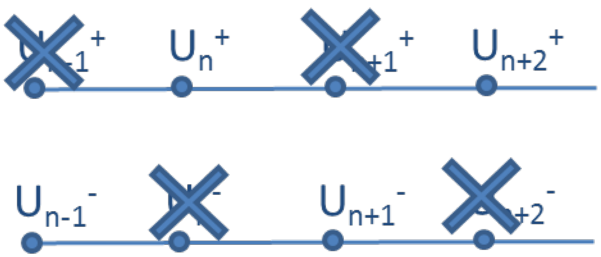


- Sum of both modes is not possible
- Total flux alternates between $\Pi_H^+(k)$ and $\Pi_H^-(k)$
- Dissipation effects cannot cancel

Ditlevsen & Giulani (2001)

HD GOY model



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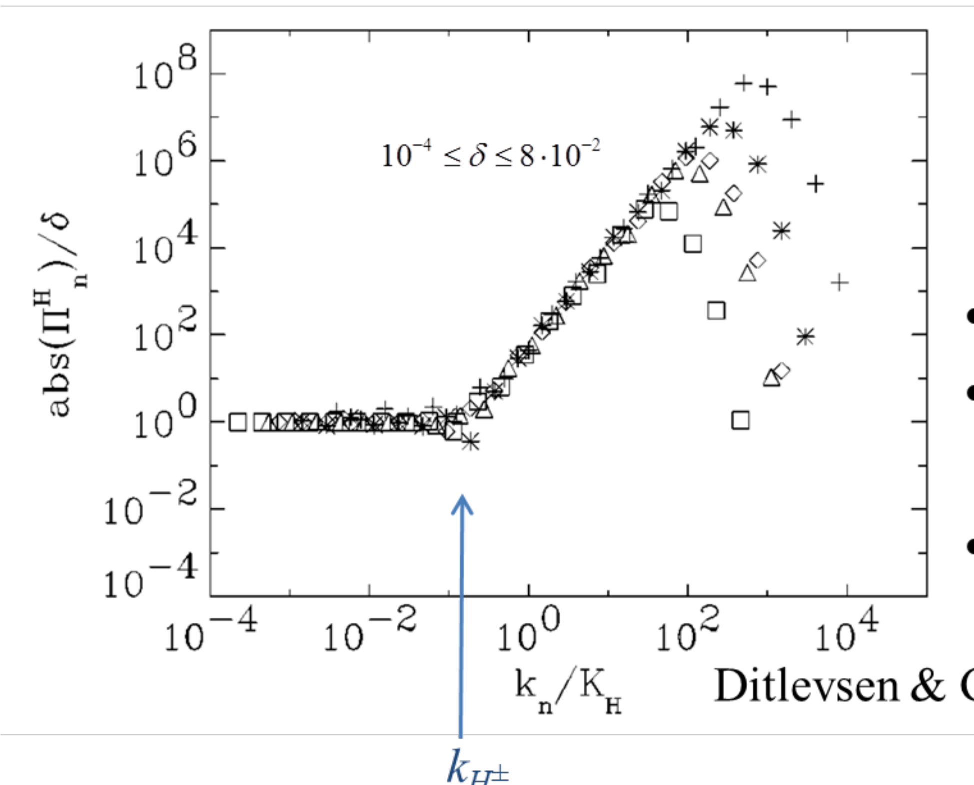
Dissipation effect => $k_{H^\pm} < k_\nu$



Ditlevsen & Giulani (2001)

- Sum of both modes is not possible
- Total flux alternates between $\Pi_H^{+}(k)$ and $\Pi_H^{-}(k)$
- Dissipation effects cannot cancel

$\Rightarrow$ No access to k_H

Conclusion

- For 2D turbulence GOY model are OK

- For 3D turbulence helical models are better to account for helicities (spectra, flux, etc)

- For both 2D and 3D, ± helical models are richer, and have a better behavior in presence of applied field or rotation

=> Why not using a RMHD ± helical model for Solar wind turbulence ?

Thank you